Statistics 351 Midterm #1 – October 10, 2007

This exam has 4 problems and 6 numbered pages.

You have 50 minutes to complete this exam. Please read all instructions carefully, and check your answers. Show all work neatly and in order, and clearly indicate your final answers. Answers must be justified whenever possible in order to earn full credit. ***Unless otherwise specified, no credit will be given for unsupported answers, even if your final answer is correct.*** *Points will be deducted for incoherent, incorrect, and/or irrelevant statements.*

This exam is closed-book, except that one $8\frac{1}{2} \times 11$ *double-sided page of handwritten notes is permitted. No other aids are allowed.*

You must answer all of the questions in the space provided. Note that blank space is NOT an indication of a question's difficulty.

Name: ______________________________

Instructor: Michael Kozdron

Problem	Score
1	
2	
3	
4	

TOTAL: __________

1. (*22 points*) Suppose that the random vector (X, Y) is jointly distributed with joint density function

$$f_{X,Y}(x,y) = \begin{cases} 2e^{-x-y}, & \text{if } 0 < x < y < \infty, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Determine the marginal density functions $f_X(x)$ and $f_Y(y)$.

(b) Based on your answer to **(a)**, are X and Y independent random variables? Justify your answer.

(c) Calculate $f_{Y|X=x}(y)$, the conditional density function of Y given $X = x$.

(continued)

Recall that the joint density function of (X, Y) is $f_{X,Y}(x, y) = \begin{cases} 2e^{-x-y}, & \text{if } 0 < x < y < \infty, \\ 0, & \text{otherwise.} \end{cases}$

(d) Use the result of **(c)** to determine $E(Y|X)$.

(e) Use the result of **(d)** to calculate $E(Y)$.

(continued)

Recall that the joint density function of (X, Y) is $f_{X,Y}(x, y) = \begin{cases} 2e^{-x-y}, & \text{if } 0 < x < y < \infty, \\ 0, & \text{otherwise.} \end{cases}$

(f) Show that $X+Y \in \Gamma(2, 1)$; that is, show that the random variable $X+Y$ has a gamma distribution with parameters $p = 2$ and $a = 1$.

2. (*8 points*) Suppose that $Y_1, Y_2, \ldots$ are independent and identically distributed random variables with $\mathbb{E}(Y_1) = 1$. Suppose further that $X_0 = 1$ and for $n = 1, 2, \ldots$, let

$$X_n = Y_1 \cdot Y_2 \cdots Y_n = \prod_{j=1}^{n} Y_j.$$

(a) Carefully verify that $\{X_n, n = 1, 2, \ldots\}$ is a martingale.

(b) Compute $\mathbb{E}(X_n)$, $n = 1, 2, \ldots$.

3. (*12 points*) Suppose that A is a continuous random variable which is uniformly distributed on $(0, 1)$ so that the density function of A is $f_A(a) = 1$, $0 < a < 1$. Suppose further that the conditional probability mass function of $X|A = a$ is

$$P(X = 1|A = a) = a \quad \text{and} \quad P(X = 0|A = a) = 1 - a.$$

(a) Determine the probability mass function of X; in other words, compute $P(X = 0)$ and $P(X = 1)$

(b) Compute the conditional mass function of A given $X = x$ for $x = 0$, 1; that is, compute $f_{A|X=0}(a)$ and $f_{A|X=1}(a)$.

4. (*8 points*) Suppose that X and Y are independent $N(0,1)$ random variables. Determine the conditional distribution of $\sqrt{3}\,X + Y$ given $X - \sqrt{3}\,Y = v$ where $v \in \mathbb{R}$.

Hint: Let $U = \sqrt{3}\,X + Y$ and let $V = X - \sqrt{3}\,Y$. Determine the distribution of $U|V = v$.