Statistics 351 (Fall 2008)
Some Computations with the Poisson Process

The *Poisson process* is a continuous time stochastic process $\{X_t, t \geq 0\}$ satisfying the following properties.

- The increments $\{X_{t_k} - X_{t_{k-1}}, k = 1, \dots, n\}$ are independent for all $0 \leq t_0 \leq t_1 \leq \cdots \leq t_n < \infty$ and all n;
- $X_0 = 0$ and there exists a $\lambda > 0$ such that

$$X_t - X_s \in \text{Po}(\lambda(t-s))$$

for $0 \leq s < t$.

Example. Suppose that $\{X_t, t \geq 0\}$ is a Poisson process with intensity λ. Let $0 < t_1 < t_2 < t_3 < t_4$. Determine

(a) $\text{Cov}(X_{t_2} - X_{t_1}, X_{t_4} - X_{t_3})$, and

(b) $\text{Cov}(X_{t_3} - X_{t_1}, X_{t_4} - X_{t_2})$.

Solution. Recall that $X_t - X_s \in \text{Po}(\lambda(t-s))$ and that disjoint increments are independent. It therefore follows that $X_{t_2} - X_{t_1}$ and $X_{t_4} - X_{t_3}$ are independent. Hence, we see that $\text{Cov}(X_{t_2} - X_{t_1}, X_{t_4} - X_{t_3}) = 0$ establishing **(a)**. As for **(b)**, we notice that $X_{t_3} - X_{t_1}$ and $X_{t_4} - X_{t_2}$ are non-disjoint increments. The trick is to notice that

$$X_{t_4} - X_{t_2} = X_{t_4} - X_{t_3} + X_{t_3} - X_{t_2}$$

which gives

$$\begin{aligned}
\text{Cov}(X_{t_3} - X_{t_1}, X_{t_4} - X_{t_2}) &= \text{Cov}(X_{t_3} - X_{t_1}, X_{t_4} - X_{t_3} + X_{t_3} - X_{t_2}) \\
&= \text{Cov}(X_{t_3} - X_{t_1}, X_{t_4} - X_{t_3}) + \text{Cov}(X_{t_3} - X_{t_1}, X_{t_3} - X_{t_2}) \\
&= 0 + \text{Cov}(X_{t_3} - X_{t_1}, X_{t_3} - X_{t_2})
\end{aligned}$$

which follows as in **(a)** since $X_{t_3} - X_{t_1}$ and $X_{t_4} - X_{t_3}$ are disjoint increments. Similarly,

$$\begin{aligned}
\text{Cov}(X_{t_3} - X_{t_1}, X_{t_3} - X_{t_2}) &= \text{Cov}(X_{t_3} - X_{t_2} + X_{t_2} - X_{t_1}, X_{t_3} - X_{t_2}) \\
&= \text{Cov}(X_{t_3} - X_{t_2}, X_{t_3} - X_{t_2}) + \text{Cov}(X_{t_2} - X_{t_1}, X_{t_3} - X_{t_2}) \\
&= \text{Var}(X_{t_3} - X_{t_2}) + 0.
\end{aligned}$$

Since $X_{t_3} - X_{t_2} \in \text{Po}(\lambda(t_3 - t_2))$, we know

$$\text{Var}(X_{t_3} - X_{t_2}) = \lambda(t_3 - t_2).$$

That is,

$$\text{Cov}(X_{t_3} - X_{t_1}, X_{t_4} - X_{t_2}) = \lambda(t_3 - t_2).$$

Example. Suppose that $\{X_t, t \geq 0\}$ is a Poisson process with intensity 1. Compute

(a) $P\{X_5 - X_1 = i | X_4 - X_1 = 2\}$ for $i = 3, 4, 5$, and

(b) $P\{X_2 - X_1 = j | X_4 - X_1 = 2\}$ for $j = 0, 1, 2$.

Solution. (a)

$$\begin{aligned}
P\{X_5 - X_1 = i | X_4 - X_1 = 2\} &= \frac{P\{X_5 - X_1 = i, X_4 - X_1 = 2\}}{P\{X_4 - X_1 = 2\}} \\
&= \frac{P\{X_5 - X_4 = i - 2, X_4 - X_1 = 2\}}{P\{X_4 - X_1 = 2\}} \\
&= \frac{P\{X_5 - X_4 = i - 2\}P\{X_4 - X_1 = 2\}}{P\{X_4 - X_1 = 2\}} \\
&= P\{X_5 - X_4 = i - 2\}.
\end{aligned}$$

Since $X_5 - X_4 \in \text{Po}(1)$, we conclude

$$P\{X_5 - X_4 = i - 2\} = \frac{1}{(i-2)!}e^{-1}, \quad i = 3, 4, 5.$$

(b)

$$\begin{aligned}
P\{X_2 - X_1 = j | X_4 - X_1 = 2\} &= \frac{P\{X_2 - X_1 = j, X_4 - X_1 = 2\}}{P\{X_4 - X_1 = 2\}} \\
&= \frac{P\{X_4 - X_2 = 2 - j, X_2 - X_1 = j\}}{P\{X_4 - X_1 = 2\}} \\
&= \frac{P\{X_4 - X_2 = 2 - j\}P\{X_2 - X_1 = j\}}{P\{X_4 - X_1 = 2\}}
\end{aligned}$$

Since $X_4 - X_2 \in \text{Po}(2)$, we find

$$P\{X_4 - X_2 = 2 - j\} = \frac{2^{2-j}}{(2-j)!}e^{-2}.$$

Since $X_2 - X_1 \in \text{Po}(1)$, we find

$$P\{X_2 - X_1 = j\} = \frac{1}{j!}e^{-1}.$$

Since $X_4 - X_1 \in \text{Po}(3)$, we find

$$P\{X_4 - X_1 = 2\} = \frac{3^2}{2!}e^{-3}.$$

Hence, combining everything gives

$$P\{X_2 - X_1 = j | X_4 - X_1 = 2\} = \frac{2^{2-j}}{(2-j)!} \cdot \frac{1}{j!} \cdot \frac{2!}{3^2} = \binom{2}{j} 2^{2-j} 3^{-2}.$$

Example. Suppose that $\{X_t, t \geq 0\}$ is a Poisson process with intensity 1. Compute

(c) $\operatorname{Var}(X_3|X_1 = x)$, and

(d) $\mathbb{E}(X_3|X_1 = x)$.

Solution. (c) Observe that X_3 and X_1 are not independent. However, $X_3 - X_1$ and X_1 *are* independent. Therefore,

$$\begin{aligned}\operatorname{Var}(X_3|X_1 = x) = \operatorname{Var}(X_3 - X_1 + X_1|X_1 = x) &= \operatorname{Var}(X_3 - X_1|X_1 = x) + \operatorname{Var}(X_1|X_1 = x) \\ &= \operatorname{Var}(X_3 - X_1)\end{aligned}$$

since $\operatorname{Var}(X_1|X_1 = x) = 0$. Using the fact that $X_3 - X_1 \in \operatorname{Po}(2)$ we conclude $\operatorname{Var}(X_3 - X_1) = 2$ and so

$$\operatorname{Var}(X_3|X_1 = x) = 2.$$

(d) Similarly, we find

$$\begin{aligned}\mathbb{E}(X_3|X_1 = x) = \mathbb{E}(X_3 - X_1 + X_1|X_1 = x) &= \mathbb{E}(X_3 - X_1|X_1 = x) + \mathbb{E}(X_1|X_1 = x) \\ &= \mathbb{E}(X_3 - X_1) + x.\end{aligned}$$

Using the fact that $X_3 - X_1 \in \operatorname{Po}(2)$ we conclude $\mathbb{E}(X_3 - X_1) = 2$ and so

$$\mathbb{E}(X_3|X_1 = x) = 2 + x.$$