Statistics 351 (Fall 2008)
Simple Random Walk

Suppose that $Y_1, Y_2, \ldots$ are i.i.d. random variables with $P\{Y_1 = 1\} = P\{Y_1 = -1\} = 1/2$, and define the *discrete time stochastic process* $\{S_n, n = 0, 1, \ldots\}$ by setting $S_0 = 0$ and

$$S_n = \sum_{i=1}^{n} Y_i.$$

We call $\{S_n, n = 0, 1, \ldots\}$ a *simple random walk* (SRW).

One useful way to visualize a SRW is to graph its *trajectory* $n \mapsto S_n$. In other words, plot the pairs of points (n, S_n), $n = 0, 1, 2, \ldots$ and join the dots with straight line segments.

(a) Suppose that a realization of the sequence $Y_1, Y_2, \ldots$ produced

$$-1, -1, -1, 1, 1, -1, 1, 1, -1, 1, 1, 1, -1, 1, 1.$$

Sketch a graph of the resulting trajectory $n \mapsto S_n$ of the SRW.

The theory of martingales is extremely useful for analyzing stochastic processes. Recall that a stochastic process $\{X_n, n = 0, 1, 2, \ldots\}$ is called a martingale if $\mathbb{E}(X_{n+1}|X_n) = X_n$ for all n.

(b) Review the proof that both S_n and $S_n^2 - n$ are martingales.

(c) Use the fact that $\mathbb{E}(X_n) = \mathbb{E}(X_0)$ for any martingale to compute $\mathbb{E}(S_n^2)$.

Even though the underlying sequence $Y_1, Y_2, \ldots$ is comprised of independent random variables, the sequence $S_1, S_2, S_3, \ldots$ is comprised of *dependent* random variables.

(d) Compute $\text{cov}(S_n, S_{n+1})$, $n = 1, 2, \ldots$.

The *transition probabilities* describe the probability that the process is at a given position at a given time. One quantity of interest is the probability that the SRW is back at the origin at time n; that is, $P\{S_n = 0\}$. Notice that the SRW can only be at the origin after an even number of steps since the only way for it to be at 0 is for there to have been an equal number of steps to the left as to the right. This means it is notationally easier to work with $P\{S_{2n} = 0\}$, $n = 0, 1, 2, \ldots$.

(e) Determine an expression for $P\{S_{2n} = 0\}$, $n = 0, 1, 2, \ldots$.

(f) Try and determine an expression for $P\{S_{2n} = x\}$ where $|x| \le 2n$ has even parity. (That is, x is an even integer between $-2n$ and $2n$.)