Statistics 351: End of Semester Course Announcements

On the STAT 351 Final Exam Information handout, I incorrectly listed Sections III.4 and V.4 as examinable. Both of these sections discuss characteristic functions and are NOT included on your final exam.

The final exam will be typeset on legal-sized paper. There are 11 questions on 12 pages, and the exam is worth a total of 150 points.

Previous final exams were also typeset on legal-sized paper. However, when I posted them online, they were automatically reformated for letter-sized paper. As such, the amount of blank space between problems is not correct.

The final exam will include a question identical to #3 on Assignment #9. The wording on the final exam is as follows. "Determine the standard form of the level curves of the density function $f_{\mathbf{X}}(x_1, x_2)$. Normalize your quadratic form so that it is in standard form, i.e., so that $Q(\mathbf{x}) = Q(\mathbf{y}) = 1$. Express your answer analytically, geometrically, and descriptively."

As noted in Stewart, the level curves of a function f of two variables are the curves with equation $f(x_1, x_2) = k$, where k is a constant in the range of f.

Since the density function for a multivariate normal $\mathbf{X} \in \mathcal{N}(\mathbf{0}, \mathbf{\Lambda})$ is

$$f_{\mathbf{X}}(x_1, x_2) = \frac{1}{2\pi\sqrt{\det[\mathbf{\Lambda}]}} \exp\left\{-\frac{1}{2}\mathbf{x}'\mathbf{\Lambda}^{-1}\mathbf{x}\right\},$$

the level curves of f are given by

$$\frac{1}{2\pi\sqrt{\det[\mathbf{\Lambda}]}} \exp\left\{-\frac{1}{2}\mathbf{x}'\mathbf{\Lambda}^{-1}\mathbf{x}\right\} = k$$

where k is an arbitrary constant. (Technically, the range of f is $0 < f(x_1, x_2) \le \frac{1}{2\pi\sqrt{\det[\mathbf{\Lambda}]}}$ so that k must be chosen in this range.)

However, we can simplify this to conclude that

$$-2\log\left(2\pi k\sqrt{\det[\mathbf{\Lambda}]}\right) = \mathbf{x}'\mathbf{\Lambda}^{-1}\mathbf{x}.$$

Now, since k is an arbitrary constant the constant $-2\log\left(2\pi k\sqrt{\det[\mathbf{\Lambda}]}\right)$ is also an arbitrary constant. Call it K. The restriction on k implies that the allowable values of K are $K \ge 0$.

Therefore, the level curves are (the infinite collection of curves) described by

$$\{\mathbf{x}'\mathbf{\Lambda}^{-1}\mathbf{x} = K \,:\, K \ge 0\}.$$

As shown in class, the equation $\mathbf{x}'\mathbf{\Lambda}^{-1}\mathbf{x} = K$ describes an ellipse centred at the origin. The eigenvalues and eigenvectors of $\mathbf{\Lambda}$ determine the length of the major and minor axes, as well as the angle of rotation.

Hence, the level curves are given by an infinite family of concentric ellipses. Thus, it is sufficient to describe ONE of them. I am telling you to choose the ellipse described by $\mathbf{x}'\mathbf{\Lambda}^{-1}\mathbf{x} = 1$. In some textbooks, this is called the standard form of the level curves. In other words, choose the value of the constant K so that $K = 1$.

Finally, it is worth noting that $\mathbf{x}'\mathbf{\Lambda}^{-1}\mathbf{x}$ is the quadratic form we called $Q(\mathbf{x})$. If we then define the new random vector $\mathbf{Y} = C'\mathbf{X}$ where C is the orthogonal matrix of eigenvectors, then $Q(\mathbf{y}) = Q(\mathbf{x})$. The advantage to this formulation is that the since $\mathbf{Y} \in \mathcal{N}(\mathbf{0}, D)$, the components of $\mathbf{Y}$ are independent. Thus, the quadratic form $Q(\mathbf{y})$ has an easy form, namely

$$Q(\mathbf{y}) = \frac{y_1^2}{\lambda_1} + \frac{y_2^2}{\lambda_2}.$$

Setting $Q(\mathbf{y}) = 1$ gives the equation of the ellipse

$$\frac{y_1^2}{\lambda_1} + \frac{y_2^2}{\lambda_2} = 1$$

which can be described as in high school.

Furthermore, the orthogonal matrix C describes the angle of rotation when viewing this ellipse back in the x_1, x_2-plane.