Statistics 351–Intermediate Probability
Fall 2008 (200830)
Final Exam Solutions

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1. (a) We see that $f_{X,Y}(x,y) \geq 0$ for all $0 < x, y < 1$, and that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_0^1\int_0^y 10xy^2\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 5y^4\,\mathrm{d}y = y^5\Big|_0^1 = 1.$$

Thus, $f_{X,Y}$ is a legitimate density.

1. (b) We compute

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,\mathrm{d}y = \int_x^1 10xy^2\,\mathrm{d}y = \frac{10x(1-x^3)}{3}, \quad 0 < x < 1.$$

1. (c) We compute

$$E(X) = \int_{-\infty}^{\infty} x f_X(x)\,\mathrm{d}x = \int_0^1 x\cdot\frac{10x(1-x^3)}{3}\,\mathrm{d}x = \frac{10}{9} - \frac{10}{18} = \frac{5}{9}.$$

1. (d) We compute

$$f_{Y|X=x}(y) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{10xy^2}{\frac{10x(1-x^3)}{3}} = \frac{3y^2}{1-x^3}, \quad x < y < 1.$$

1. (e) We compute

$$E(Y|X=x) = \int_{-\infty}^{\infty} y f_{Y|X=x}(y)\,\mathrm{d}y = \int_x^1 y\cdot\frac{3y^2}{1-x^3}\,\mathrm{d}y = \frac{3(1-x^4)}{4(1-x^3)}.$$

1. (f) Using properties of conditional expectation (Theorem II.2.1), we compute

$$E(Y) = E(\,E(Y|X)\,) = E\left(\frac{3(1-X^4)}{4(1-X^3)}\right) = \int_0^1 \frac{3(1-x^4)}{4(1-x^3)}\cdot\frac{10x(1-x^3)}{3}\,\mathrm{d}x = \frac{5}{2}\int_0^1 x - x^5\,\mathrm{d}x = \frac{5}{6}.$$

1. (g) If $U = XY$ and $V = Y$, then solving for X and Y gives $X = U/V$ and $Y = V$, so that the Jacobian of this transformation is

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1/v & -u/v^2 \\ 0 & 1 \end{vmatrix} = \frac{1}{v}.$$

By Theorem I.2.1, the joint density of $(U,V)'$ is therefore given by

$$f_{U,V}(u,v) = f_{X,Y}(u/v, v)\cdot|J| = 10uv^{-1}v^2\cdot v^{-1} = 10u$$

provided that $0 < u < v^2 < 1$ and $v > 0$. It then follows that the density function of U is given by

$$f_U(u) = \int_{-\infty}^{\infty} f_{U,V}(u,v)\,\mathrm{d}v = \int_{\sqrt{u}}^1 10u\,\mathrm{d}v = 10u(1-\sqrt{u})$$

provided that $0 < u < 1$.

2. (a) The density function of X is given by

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_{-\infty}^{\infty} f_{X|Y=y}(x) \cdot f_Y(y)\,\mathrm{d}y.$$

Substituting in gives

$$f_X(x) = \int_0^{\infty} \theta y x^{\theta-1} \exp\{-yx^{\theta}\} \cdot \frac{a^p}{\Gamma(p)} y^{p-1} e^{-ay}\,\mathrm{d}y = \frac{a^p}{\Gamma(p)} \theta x^{\theta-1} \int_0^{\infty} y^p \exp\left\{-y(a+x^{\theta})\right\}\,\mathrm{d}y.$$

Properties of the gamma function imply that

$$\int_0^{\infty} y^p \exp\left\{-y(a+x^{\theta})\right\}\,\mathrm{d}y = \Gamma(p+1) \cdot (a+x^{\theta})^{-(p+1)}$$

so that

$$f_X(x) = \frac{a^p}{\Gamma(p)} \theta x^{\theta-1} \cdot \Gamma(p+1) \cdot (a+x^{\theta})^{-(p+1)} = \frac{p\,a^p\,\theta\,x^{\theta-1}}{(a+x^{\theta})^{p+1}}$$

provided that $x > 0$ (and using the fact that $\Gamma(p+1) = p\Gamma(p)$.)

2. (b) If $Z = X^{\theta}$, then the density function of Z is

$$f_Z(z) = f_X(z^{1/\theta}) \cdot \frac{\mathrm{d}}{\mathrm{d}z} z^{1/\theta} = \frac{p\,a^p\,\theta\,z^{1-1/\theta}}{(a+z)^{p+1}} \cdot \frac{1}{\theta} z^{1/\theta-1} = \frac{p\,a^p}{(a+z)^{p+1}}$$

provided that $z > 0$.

3. There are two ways to compute the resulting integral, depending on the order of integration. One way is much easier than the other.

Solution 1: $(\mathrm{d}y\,\mathrm{d}x)$
We begin by noticing that

$$P\{X < Y\} = \iint_{\{x<y\}} f_{X,Y}(x,y)\,\mathrm{d}y\,\mathrm{d}x = \int_0^{\infty} \int_x^{\infty} \frac{x}{27} \exp\left\{-\frac{x+y}{3}\right\}\,\mathrm{d}y\,\mathrm{d}x$$

$$= \int_0^{\infty} \frac{x}{27} e^{-x/3} \int_x^{\infty} e^{-y/3}\,\mathrm{d}y\,\mathrm{d}x.$$

We now find

$$\int_x^{\infty} e^{-y/3}\,\mathrm{d}y = 3e^{-x/3}$$

which implies that

$$P\{X < Y\} = \int_0^{\infty} \frac{x}{9} e^{-2x/3}\,\mathrm{d}x.$$

We recognize this as a Gamma function; that is,

$$\int_0^{\infty} \frac{x}{9} e^{-2x/3}\,\mathrm{d}x = \frac{1}{4} \int_0^{\infty} \frac{4x}{9} e^{-2x/3}\,\mathrm{d}x = \frac{1}{4} \int_0^{\infty} u e^{-u}\,\mathrm{d}u = \frac{1}{4}\Gamma(2) = \frac{1}{4}.$$

Solution 2: $(\mathrm{d}x\,\mathrm{d}y)$
In this case, we find

$$P\{X<Y\} = \iint_{\{x<y\}} f_{X,Y}(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_0^\infty \int_0^y \frac{x}{27}\exp\left\{-\frac{x+y}{3}\right\}\,\mathrm{d}x\,\mathrm{d}y$$
$$= \frac{1}{3}\int_0^\infty \exp\left\{-\frac{y}{3}\right\}\int_0^y \frac{x}{9}\exp\left\{-\frac{x}{3}\right\}\,\mathrm{d}x\,\mathrm{d}y.$$

We now observe (using the integration-by-parts formula given on the first page with $a=1/3$) that

$$\int_0^y \frac{x}{9}\exp\left\{-\frac{x}{3}\right\}\,\mathrm{d}x = \left(-\frac{x}{3}e^{-x/3} - e^{-x/3}\right)\Big|_0^y = 1 - \frac{y}{3}e^{-y/3} - e^{-y/3}.$$

Substituting back in gives

$$\begin{aligned} P\{X<Y\} &= \frac{1}{3}\int_0^\infty e^{-y/3}\left(1-\frac{y}{3}e^{-y/3} - e^{-y/3}\right)\,\mathrm{d}y \\ &= \frac{1}{3}\int_0^\infty e^{-y/3} - \frac{y}{3}e^{-2y/3} - e^{-2y/3}\,\mathrm{d}y \\ &= \int_0^\infty \frac{1}{3}e^{-y/3} - \frac{y}{9}e^{-2y/3} - \frac{1}{3}e^{-2y/3}\,\mathrm{d}y \end{aligned}$$

The easiest way to calculate these integrals is to recognize them as densities. That is,

$$\int_0^\infty \frac{1}{3}e^{-y/3}\,\mathrm{d}y = 1, \qquad \int_0^\infty \frac{1}{3}e^{-2y/3}\,\mathrm{d}y = \frac{1}{2}\int_0^\infty \frac{2}{3}e^{-2y/3}\,\mathrm{d}y = \frac{1}{2},$$

and

$$\int_0^\infty \frac{y}{9}e^{-2y/3}\,\mathrm{d}y = \frac{1}{4}\int_0^\infty \frac{4y}{9}e^{-2y/3}\,\mathrm{d}y = \frac{1}{4}.$$

Thus, we finally conclude that

$$P\{X<Y\} = 1 - \frac{1}{2} - \frac{1}{4} = \frac{1}{4}.$$

4. (a) The distribution of X is $U(0,1)$ so that

$$f_X(x) = 1, \quad 0<x<1,$$

and the conditional distribution of $Y|X=x$ is $U(x,1)$ so that

$$f_{Y|X=x}(y) = \frac{1}{1-x}, \quad x<y<1.$$

Therefore, the joint density function of $(X,Y)'$ is

$$f_{X,Y}(x,y) = f_{Y|X=x}(y)\cdot f_X(x) = \frac{1}{1-x}$$

provided that $0<x<y<1$.

4. (b) Using Corollary II.2.3.1 on page 39, we know

$$\mathrm{Var}(Y) = E(\mathrm{Var}(Y|X)) + \mathrm{Var}(E(Y|X)).$$

Since $Y|X = x$ is $U(x, 1)$ we know that $E(Y|X = x) = (1 + x)/2$ and $\mathrm{Var}(Y|X = x) = (1 - x)^2/12$ so that

$$\begin{aligned}\mathrm{Var}(Y) = E(\mathrm{Var}(Y|X)) + \mathrm{Var}(E(Y|X)) &= E\left(\frac{(1-X)^2}{12}\right) + \mathrm{Var}\left(\frac{1+X}{2}\right)\\ &= \frac{1}{12} - \frac{E(X)}{6} + \frac{E(X^2)}{12} + \frac{\mathrm{Var}(X)}{4}.\end{aligned}$$

Since X is $U(0, 1)$, we know that $E(X) = 1/2$ and $\mathrm{Var}(X) = 1/12$ so that $E(X^2) = 1/3$. Thus, we conclude that

$$\mathrm{Var}(Y) = \frac{1}{12} - \frac{1}{12} + \frac{1}{36} + \frac{1}{48} = \frac{7}{144}.$$

5. (a) Since X_1 and X_2 are independent and identically distributed $\Gamma(2, 1)$ random variables, they have common density function

$$f(x) = xe^{-x}, \quad x > 0,$$

and common distribution function

$$F(x) = \int_0^x ue^{-u}\,\mathrm{d}u = 1 - xe^{-x} - e^{-x}$$

for $x > 0$ (and $F(x) = 0$ for $x \le 0$) using the integration-by-parts formula given on the first page. Therefore, it follows from Theorem IV.1.2 that the density function of $X_{(1)}$ is

$$f_{X_{(1)}}(y) = 2(1 - F(y))f(y) = 2ye^{-y}(ye^{-y} + e^{-y}) = 2y(y + 1)e^{-2y}$$

provided that $y > 0$.

5. (b) It follows from Theorem IV.1.2 that the density function of $X_{(2)}$ is

$$f_{X_{(2)}}(y) = 2F(y)f(y) = 2ye^{-y}(1 - ye^{-y} - e^{-y})$$

provided that $y > 0$.

5. (c) It follows from Theorem IV.2.1 that the density function of $(X_{(1)}, X_{(2)})'$ is

$$f_{X_{(1)},X_{(2)}}(y_1, y_2) = 2f(y_1)f(y_2) = 2y_1y_2e^{-y_1-y_2}$$

provided that $0 < y_1 < y_2 < \infty$.

5. (d) If $U = X_{(1)}/X_{(2)}$ and $V = X_{(2)}$, then solving for $X_{(1)}$ and $X_{(2)}$ gives $X_{(1)} = UV$ and $X_{(2)} = V$ so that the Jacobian of this transformation is

$$J = \begin{vmatrix} \frac{\partial y_1}{\partial u} & \frac{\partial y_1}{\partial v} \\ \frac{\partial y_2}{\partial u} & \frac{\partial y_2}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ 0 & 1 \end{vmatrix} = v.$$

By Theorem I.2.1, the joint density of $(U, V)'$ is therefore given by

$$f_{U,V}(u, v) = f_{X_{(1)},X_{(2)}}(uv, v) \cdot |J| = 2uv^3e^{-uv-v}$$

provided that $0 < u < 1$ and $0 < v < \infty$.

It now follows that the density function of U is

$$f_U(u) = \int_{-\infty}^{\infty} f_{U,V}(u,v)\,\mathrm{d}v = \int_0^{\infty} 2uv^3 e^{-uv-v}\,\mathrm{d}v = 2u\int_0^{\infty} v^3 e^{-v(u+1)}\,\mathrm{d}v = 2u\cdot\Gamma(4)(u+1)^{-4}$$
$$= \frac{12u}{(1+u)^4}$$

provided that $0 < u < 1$ and using properties of the gamma function.

6. (a) In order to find the eigenvalues of $\mathbf{\Lambda}$, we must find those values of λ such that $\det[\mathbf{\Lambda} - \lambda I] = 0$. Therefore,

$$\det[\mathbf{\Lambda} - \lambda I] = \det\begin{bmatrix} \frac{3}{2} - \lambda & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{5}{2} - \lambda \end{bmatrix} = \left(\frac{3}{2} - \lambda\right)\left(\frac{5}{2} - \lambda\right) - \frac{3}{4} = \lambda^2 - 4\lambda + 3 = (\lambda - 1)(\lambda - 3)$$

so that the eigenvalues of $\mathbf{\Lambda}$ are $\lambda_1 = 1$ and $\lambda_2 = 3$.

6. (b) Since $\lambda_1 = 1$,

$$[\mathbf{\Lambda} - \lambda_1 I \,|\, \mathbf{0}] = \left[\begin{array}{cc|c} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & \frac{3}{2} & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -\sqrt{3} & 0 \\ 0 & 0 & 0 \end{array}\right],$$

and since $\lambda_2 = 3$,

$$[\mathbf{\Lambda} - \lambda_2 I \,|\, \mathbf{0}] = \left[\begin{array}{cc|c} -\frac{3}{2} & -\frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} -\frac{3}{2} & -\frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} -3 & -\sqrt{3} & 0 \\ 0 & 0 & 0 \end{array}\right],$$

we conclude that eigenvectors for λ_1 and λ_2 are

$$\mathbf{v}_1 = \begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ \sqrt{3} \end{bmatrix},$$

respectively. Therefore, the diagonal matrix is

$$D = \mathrm{diag}(\lambda_1, \lambda_2) = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

and the orthogonal matrix is

$$C = \left[\begin{array}{cc} \frac{\mathbf{v}_1}{||\mathbf{v}_1||} & \frac{\mathbf{v}_2}{||\mathbf{v}_2||} \end{array}\right] = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

since $||\mathbf{v}_1|| = ||\mathbf{v}_2|| = 2$.

6. (c) Since $\det[\mathbf{\Lambda}] = 3$, we see that

$$\mathbf{\Lambda}^{-1} = \begin{bmatrix} \frac{5}{6} & -\frac{\sqrt{3}}{6} \\ -\frac{\sqrt{3}}{6} & \frac{1}{2} \end{bmatrix}.$$

Therefore, the density function of $\mathbf{X}$ is

$$f_{\mathbf{X}}(x_1, x_2) = \frac{1}{2\pi\sqrt{3}} \exp\left\{-\frac{1}{2}\left(\frac{5}{6}x_1^2 + \frac{\sqrt{3}}{3}x_1x_2 + \frac{1}{2}x_2^2\right)\right\}$$

provided that $-\infty < x_1, x_2 < \infty$.

6. (d) Since $X_1 \in \mathcal{N}(0, 3/2)$ we see that

$$f_{X_1}(0) = \frac{\sqrt{2}}{\sqrt{2\pi}\sqrt{3}}$$

so that, by definition,

$$f_{X_2|X_1=0}(x_2) = \frac{f_{\mathbf{X}}(0, x_2)}{f_{X_1}(0)} = \frac{\frac{1}{2\pi\sqrt{3}} \exp\left\{-\frac{1}{2}\left(\frac{1}{2}x_2^2\right)\right\}}{\frac{\sqrt{2}}{\sqrt{2\pi}\sqrt{3}}} = \frac{1}{\sqrt{2}\sqrt{2\pi}} \exp\left\{-\frac{1}{4}x_2^2\right\}.$$

Thus, we conclude that $X_2|X_1 = 0$ has a $\mathcal{N}(0, 2)$ distribution.

6. (e) If $\mathbf{Y} = C'\mathbf{X}$, then by Theorem V.3.1, $\mathbf{Y}$ is MVN with mean $C'\boldsymbol{\mu}$ and covariance matrix $C'\mathbf{\Lambda}C'' = C'\mathbf{\Lambda}C = D$ using our result from **(b)**. Hence, we conclude

$$\mathbf{Y} \in N\left(\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}1 & 0\\0 & 3\end{bmatrix}\right).$$

6. (f) Since $\mathbf{Y}$ is multivariate normal we know from Definition I that Y_1 and Y_2 are each one-dimensional normals. We also know from Theorem V.7.1 that the components of $\mathbf{Y}$ are independent if and only if they are uncorrelated. From **(e)** we know that $\mathrm{Cov}(Y_1, Y_2) = 0$ so that Y_1 and Y_2 are, in fact, independent.

6. (g) We know from **(e)** that if $\mathbf{Y} = C'\mathbf{X}$, then $\mathbf{Y} \in \mathcal{N}(\mathbf{0}, D)$ so that $Q(\mathbf{x}) = \mathbf{x}'\mathbf{\Lambda}^{-1}\mathbf{x} = \mathbf{y}'D^{-1}\mathbf{y} = Q(\mathbf{y})$. Setting $Q(\mathbf{y}) = 1$ gives

$$\frac{y_1^2}{\lambda_1} + \frac{y_2^2}{\lambda_2} = y_1^2 + \frac{y_2^2}{3} = 1.$$

This describes an ellipse centred at the origin passing through the points $(0, \sqrt{3})'$, $(0, -\sqrt{3})'$, $(1, 0)'$, and $(-1, 0)'$. We now notice that we can write C as

$$C = \begin{bmatrix}\frac{\sqrt{3}}{2} & -\frac{1}{2}\\ \frac{1}{2} & -\frac{\sqrt{3}}{2}\end{bmatrix} = \begin{bmatrix}\cos(\pi/6) & -\sin(\pi/6)\\ \sin(\pi/6) & \cos(\pi/6)\end{bmatrix}.$$

This matrix describes a counterclockwise rotation by $\pi/6 = 30°$. Thus,

$$Q(\mathbf{x}) = \mathbf{x}'\mathbf{\Lambda}^{-1}\mathbf{x} = \frac{5}{6}x_1^2 + \frac{\sqrt{3}}{3}x_1x_2 + \frac{1}{2}x_2^2 = 1$$

describes the same ellipse rotated by $\pi/6$. In other words, it is an ellipse passing through the points

$$C\begin{bmatrix}0\\\sqrt{3}\end{bmatrix} = \left(-\frac{\sqrt{3}}{2}, \frac{3}{2}\right)', \quad C\begin{bmatrix}0\\-\sqrt{3}\end{bmatrix} = \left(\frac{\sqrt{3}}{2}, -\frac{3}{2}\right)', \quad C\begin{bmatrix}1\\0\end{bmatrix} = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)', \quad C\begin{bmatrix}-1\\0\end{bmatrix} = \left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)'.$$

7. **(a)** Let

$$B = \begin{bmatrix} 1/\sigma_1 & -\rho/\sigma_2 \\ 0 & 1/\sigma_2 \end{bmatrix}$$

so that $\mathbf{Y} = B\mathbf{X}$. By Theorem V.3.1, $\mathbf{Y}$ is MVN with mean

$$B\boldsymbol{\mu} = \begin{bmatrix} 1/\sigma_1 & -\rho/\sigma_2 \\ 0 & 1/\sigma_2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and covariance matrix

$$B\mathbf{\Lambda}B' = \begin{bmatrix} 1/\sigma_1 & -\rho/\sigma_2 \\ 0 & 1/\sigma_2 \end{bmatrix} \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix} \begin{bmatrix} 1/\sigma_1 & 0 \\ -\rho/\sigma_2 & 1/\sigma_2 \end{bmatrix} = \begin{bmatrix} 1-\rho^2 & 0 \\ 0 & 1 \end{bmatrix}.$$

7. **(b)** Since $\mathbf{Y}$ is multivariate normal we know from Definition I that Y_1 and Y_2 are each one-dimensional normals. We also know from Theorem V.7.1 that the components of $\mathbf{Y}$ are independent if and only if they are uncorrelated. From **(a)** we know that $\text{Cov}(Y_1, Y_2) = 0$ so that Y_1 and Y_2 are, in fact, independent.

8. We know from Theorem V.9.1 that $\mathbf{X}'\mathbf{\Lambda}^{-1}\mathbf{X} \in \chi^2(3)$. Thus, the required matrix A is

$$A = \mathbf{\Lambda}^{-1} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}.$$

9. If $M_n = S_n^3 - 3nS_n$, then

$$\begin{aligned} M_{n+1} = S_{n+1}^3 - 3(n+1)S_{n+1} &= (S_n + Y_{n+1})^3 - 3(n+1)(S_n + Y_{n+1}) \\ &= S_n^3 + 3S_n^2 Y_{n+1} + 3S_n Y_{n+1}^2 + Y_{n+1}^3 - 3(n+1)S_n - 3(n+1)Y_{n+1} \\ &= M_n + 3S_n(Y_{n+1}^2 - 1) + 3S_n^2 Y_{n+1} - 3(n+1)Y_{n+1} + Y_{n+1}^3. \end{aligned}$$

Thus, we see that we will be able to conclude that $\{M_n, n = 0, 1, \ldots\}$ is a martingale if we can show that

$$E\left(3S_n(Y_{n+1}^2 - 1) + 3S_n^2 Y_{n+1} - 3(n+1)Y_{n+1} + Y_{n+1}^3 | S_n\right) = 0.$$

Now

$$3E(S_n(Y_{n+1}^2 - 1)|S_n) = 3S_n E(Y_{n+1}^2 - 1) \quad \text{and} \quad 3E(S_n^2 Y_{n+1}|S_n) = 3S_n^2 E(Y_{n+1})$$

by "taking out what is known," and using the fact that Y_{n+1} and S_n are independent. Furthermore,

$$3(n+1)E(Y_{n+1}|S_n) = 3(n+1)E(Y_{n+1}) \quad \text{and} \quad E(Y_{n+1}^3|S_n) = E(Y_{n+1}^3)$$

using the fact that Y_{n+1} and S_n are independent. Since $E(Y_{n+1}) = 0$, $E(Y_{n+1}^2) = 1$, and $E(Y_{n+1}^3) = 0$, we see that

$$\begin{aligned} E(M_{n+1}|M_n) &= M_n + 3S_n E(Y_{n+1}^2 - 1) + 3S_n^2 E(Y_{n+1}) - 3(n+1)E(Y_{n+1}) + E(Y_{n+1}^3) \\ &= M_n + 3S_n \cdot (1-1) + 3S_n^2 \cdot 0 - 3(n+1) \cdot 0 + 0 \\ &= M_n \end{aligned}$$

which proves that $\{M_n, n = 0, 1, 2, \ldots\}$ is, in fact, a martingale.

10. (a) The optional sampling theorem applied to the martingale $\{S_n, n = 0, 1, 2, \ldots\}$ gives $E(S_T) = E(S_0) = 0$. Since

$$E(S_T) = -aP(S_T = -a) + bP(S_T = b) = -a[1 - P(S_T = b)] + bP(S_T = b)$$

we see that

$$-a[1 - P(S_T = b)] + bP(S_T = b) = 0$$

which solving for $P(S_T = b)$ gives

$$P(S_T = b) = \frac{a}{a+b}$$

as required.

10. (b) The optional sampling theorem applied to the martingale $\{X_n, n = 0, 1, 2, \ldots\}$ gives $E(X_T) = E(X_0) = 0$. But $E(X_T) = E(S_T^2 - T)$ and $E(X_0) = 0$ which implies that $E(S_T^2) - E(T) = 0$. Since

$$E(S_T^2) = (-a)^2 P(S_T = -a) + b^2 P(S_T = b) = a^2 \left(1 - \frac{a}{a+b}\right) + b^2 \left(\frac{a}{a+b}\right) = ab$$

using our result from **(a)**, we conclude that

$$E(T) = ab$$

as required.

11. (a) Let $\{X_t, t \geq 0\}$ denote the Poisson process with intensity 1 according to which subway trains arrive (where t is measured in units of *four minutes*). The random number of subway trains that arrive in one hour (i.e., 15 time units) is X_{15}. Since $X_t \in \text{Po}(t)$ for all t by the Poisson process assumption, we conclude that $\mathbb{E}(X_{15}) = 15$.

11. (b) If we use the fact that a Poisson process resets at fixed times, then the probability that Christian will wait at least 8 minutes (i.e., 2 time units) for the next train to arrive is

$$P(T_1 \geq 2) = P(X_2 \leq 1) = P(X_2 = 0) + P(X_2 = 1) = \frac{e^{-2}2^0}{0!} + \frac{e^{-2}2^1}{1!} = 3e^{-2}$$

using the fact that $X_2 \in \text{Po}(2)$.

11. (b) We again use the fact that a Poisson process resets at fixed times. The probability that at least three trains pass Christian in the 16 minutes (i.e., 4 time units) while he is waiting for Veronica is

$$\begin{aligned} P(X_4 \geq 3) &= 1 - P(X_4 < 3) = 1 - P(X_4 = 0) - P(X_4 = 1) - P(X_4 = 2) \\ &= 1 - \frac{e^{-4}4^0}{0!} - \frac{e^{-4}4^1}{1!} - \frac{e^{-4}4^2}{2!} \\ &= 1 - 13e^{-4}. \end{aligned}$$