Make sure that this examination has 11 numbered pages

University of Regina
Department of Mathematics & Statistics
Final Examination
200630
(December 13, 2006)

Statistics 351
Probability I

Name: ______________________ **Student Number:** ______________

Instructor: Michael Kozdron **Time:** 3 hours

Read all of the following information before starting the exam.

*You have **3** hours to complete this exam. Please read all instructions carefully, and check your answers. Show all work neatly and in order, and clearly indicate your final answers. Answers must be justified whenever possible in order to earn full credit.* ***Unless otherwise specified, no credit will be given for unsupported answers, even if your final answer is correct.***

You may use standard notation; however, any new notations or abbreviations that you introduce must be clearly defined.

*Calculators are permitted; however, you must still show all your work. You are also permitted to have **TWO** 8.5×11 pages of handwritten notes (double-sided) for your personal use. Other than these exceptions, no other aids are allowed.*

Note that blank space is not an indication of a question's difficulty. The order of the test questions is essentially random; they are not intentionally written easiest-to-hardest.

*This test has **11** numbered pages with **11** questions totalling **150** points. The number of points per question is indicated.*

Fact: For $\alpha > 0$, $\theta > 0$, the density of a random variable $X \in \Gamma(\alpha, \theta)$ is

$$f_X(x) = \frac{\theta^{-\alpha}}{\Gamma(\alpha)} x^{\alpha-1} \exp\left\{-\frac{x}{\theta}\right\}, \quad 0 < x < \infty.$$

Fact: For $p > 0$, the Gamma function is given by

$$\Gamma(p) = \int_0^\infty x^{p-1} e^{-x} \, dx.$$

DO NOT WRITE BELOW THIS LINE

Problem 1	______	Problem 2	______	Problem 3	______
Problem 4	______	Problem 5	______	Problem 6	______
Problem 7	______	Problem 8	______	Problem 9	______
Problem 10	______	Problem 11	______		
				TOTAL	______

1. (*10 points*) Let $\alpha > 0$, $\beta > 0$, $\theta > 0$, and suppose that X and Y have independent Gamma distributions with parameters (α, θ) and (β, θ), respectively. Thus, the joint density function of $(X, Y)'$ is

$$f_{X,Y}(x, y) = \frac{\theta^{-\alpha-\beta}}{\Gamma(\alpha)\Gamma(\beta)}\, x^{\alpha-1}\, y^{\beta-1} \exp\left\{-\frac{x+y}{\theta}\right\}, \quad 0 < x < \infty,\ 0 < y < \infty.$$

Consider the random variables U and V defined by $U = \dfrac{X}{X+Y}$ and $V = X + Y$.

(a) Find the joint density function of $(U, V)'$.

(b) Find the density function of U.

2. (*27 points*) Suppose that a random vector $(X, Y)'$ has joint density function

$$f_{X,Y}(x, y) = \begin{cases} 12y^2, & \text{if } 0 < y < x < 1, \\ 0, & \text{otherwise.} \end{cases}.$$

(a) Verify that $f_{X,Y}$ is, in fact, a density.

(b) Find $f_X(x)$, the marginal density function of X.

(c) Find $f_Y(y)$, the marginal density function of Y.

(d) Find $f_{X|Y=y}(x)$, the conditional density function of $X|Y = y$.

(e) Find $f_{Y|X=x}(y)$, the conditional density function of $Y|X = x$.

(f) Compute $E(X)$.

(g) Compute $E(Y|X = x)$.

(h) Use your results of **(f)** and **(g)** to compute $E(Y)$.

(i) Compute $E(XY)$.

3. (*12 points*) Let $\mathbf{X} = (X_1, X_2, X_3)'$ have the multivariate normal distribution

$$\mathbf{X} \in N\left(\begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}1 & 0 & -1\\0 & 2 & -1\\-1 & -1 & 1\end{pmatrix}\right).$$

Let the random vector $\mathbf{Y} = (Y_1, Y_2)'$ be defined by $Y_1 = X_1 - 2X_2 + X_3$ and $Y_2 = X_2 + X_3$.

(a) Determine the distribution of $\mathbf{Y}$.

(b) Determine $f_{Y_1,Y_2}(y_1, y_2)$, the density function of $\mathbf{Y}$.

4. (*12 points*) Suppose that the random vector $\mathbf{X} = (X, Y)'$ has density function

$$f_{\mathbf{X}}(x, y) = \frac{1}{4\pi} \exp\left\{ -\frac{1}{2}\left(\frac{x^2}{2} - xy + y^2 \right) \right\}.$$

(a) Determine the distribution of $\mathbf{X}$.

(b) Determine the characteristic function of $\mathbf{X}$.

(c) Determine the conditional distribution of $Y|X = x$.

5. (*8 points*) Suppose that the random variable X is normally distributed with mean 0 and variance 1. That is, $X \in N(0,1)$. Suppose further that $Y|X = x \in N(x,1)$. Determine the distribution of Y.

6. (*24 points*) Suppose that the random vector $\mathbf{X} = (X_1, X_2)'$ has the multivariate normal distribution $\mathbf{X} \in N(\boldsymbol{\mu}, \mathbf{\Lambda})$ where

$$\boldsymbol{\mu} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{\Lambda} = \begin{pmatrix} 6 & -5 \\ -5 & 6 \end{pmatrix}.$$

(a) Determine the eigenvalues of $\mathbf{\Lambda}$.

(b) Find an orthogonal matrix C and a diagonal matrix D such that $C'\mathbf{\Lambda}C = D$.

(c) Determine the distribution of $\mathbf{Y} = (Y_1, Y_2)'$ where $\mathbf{Y} = C'\mathbf{X}$. *Hint*: Use **(b)**.

(d) Explain why the random variables Y_1 and Y_2 are independent.

7. (*10 points*) Suppose that X and Y are jointly distributed with density function

$$f_{X,Y}(x,y) = 2x, \quad 0 < x < 1, \quad 0 < y < 1.$$

Compute $P(X^2 < Y < X)$

8. (*16 points*) Suppose that X_1 and X_2 are independent Exp(1) random variables. As always, let $X_{(k)}$ denote the kth order variable. Define the *sample median* by $X_M = \dfrac{X_{(2)} - X_{(1)}}{2}$ and the *sample mean* by $\overline{X} = \dfrac{X_1 + X_2}{2}$.

(a) Determine the joint density function of the sample median X_M and the sample mean $\overline{X}$.
Hint: $X_1 + X_2 = X_{(1)} + X_{(2)}$

(b) Determine the density function of the sample median X_M.

(c) Determine the density function of the sample mean $\overline{X}$.

9. (*8 points*) Suppose that the random vector $\mathbf{X} = (X, Y)'$ has the multivariate normal distribution

$$\mathbf{X} \in N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right)$$

where $\rho = \text{cov}(X, Y) > 0$.

(a) Prove that X and $Y - E(Y|X)$ are uncorrelated.

(b) Prove that X and $Y - E(Y|X)$ are independent.

10. (*15 points*) Suppose that $\{X_t,\, t \geq 0\}$ is a Poisson process with intensity $\lambda = 1$. Evaluate the following:

(a) $P(X_4 = j)$ for $j = 1, 2$;

(b) $P(X_4 = j | X_3 = 1)$ for $j = 1, 2$;

(c) $P(X_1 = 0 | X_3 = 1)$;

(d) $\text{cov}(X_3, X_4)$;

(e) $E(X_3 | X_1 = j)$ for $j = 0, 1, 2, \ldots$.

11. (*8 points*) On any given day, the number of cigarettes that Keith Richards has lit since he woke up follows a Poisson process with an intensity of $\lambda = 4$ cigarettes per hour. You may assume that Keith Richards wakes up at 10:00 a.m. every day.

(a) What is the expected time of the day at which he lights his 8th cigarette?

(b) What is the probability that he lights 3 cigarettes or more between noon and 1 p.m.?