Stat 351 Fall 2008
Assignment #8 Solutions

1. Recall from Stat 251 that if $X \in N(0,1)$, then $X^2 \in \chi^2(1)$. Furthermore, recall that if $Z_1, \ldots, Z_n$ are independent with $Z_j \in \chi^2(p_j)$, then

$$\sum_{j=1}^{n} Z_j \in \chi^2(p_1 + \cdots + p_n).$$

(That is, the sum of independent chi-squared random variables is itself chi-squared with degrees of freedom additive.) Since

$$\mathbf{X}'\mathbf{X} = X_1^2 + X_2^2 + \cdots + X_n^2$$

is the sum of n i.i.d. $\chi^2(1)$ random variables, we conclude

$$\mathbf{X}'\mathbf{X} \in \chi^2(n).$$

2. (a) Since X and Y are i.i.d. $N(0,1)$, we know that

$$3X + 4Y \in N(0, 3^2 + 4^2) = N(0, 25).$$

Normalizing implies that

$$Z = \frac{3X + 4Y}{5} \in N(0,1).$$

Thus,

$$P(3X + 4Y > 5) = P(Z > 1) \doteq 0.1587$$

using a table of normal probabilities.

2. (b) Since X and Y are independent, we know that

$$P(\min\{X,Y\} > 1) = P(X > 1, Y > 1) = P(X > 1) \cdot P(Y > 1) \doteq (0.1587)^2$$

and so

$$P(\min\{X,Y\} < 1) \doteq 1 - (0.1587)^2 \doteq 0.9748$$

using a table of normal probabilities.

2. (c) Since

$$P(|\min\{X,Y\}| < 1) = P(-1 < \min\{X,Y\} < 1) = P(\min\{X,Y\} < 1) - P(\min\{X,Y\} < -1)$$

and

$$P(\min\{X,Y\} < -1) = 1 - P(\min\{X,Y\} > -1) = 1 - P(X > -1) \cdot P(Y > -1) \doteq 1 - (0.8413)^2$$

we conclude that

$$\begin{aligned} P(|\min\{X,Y\}| < 1) &\doteq [1 - (0.1587)^2] - [1 - (0.8413)^2] = (0.8413)^2 - (0.1587)^2 \\ &\doteq 0.6826 \end{aligned}$$

using a table of normal probabilities.

2. (d) Notice that

$$\max\{X,Y\} - \min\{X,Y\} = |X - Y|$$

and that $X - Y \in N(0,2)$. Normalizing implies

$$Z = \frac{X-Y}{\sqrt{2}} \in N(0,1)$$

and so we find

$$\begin{aligned} P(\max\{X,Y\} - \min\{X,Y\} < 1) = P(|X-Y| < 1) &= P(|Z| < 1/\sqrt{2}\,) \\ &= P(-1/\sqrt{2} < Z < 1/\sqrt{2}\,) \\ &\doteq 0.5205 \end{aligned}$$

using a table of normal probabilities.

2. (e) As in Problem **1.** we note that $X^2 + Y^2 \in \chi^2(2)$. However, we know that $\chi^2(2) = \Gamma(1,2) = \text{Exp}(2)$. Thus, if $Z = X^2 + Y^2$ so that $Z \in \text{Exp}(2)$, then

$$P(X^2 + Y^2 \le 1) = P(Z \le 1) = 1 - e^{-1/2}.$$

Problem #1, page 143: Let $\mathbf{X} = (X,Y)'$ with

$$\mathbf{X} \in N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \right),$$

and consider the change of variables to polar coordinates $(R,\Theta)'$. The inverse of this transformation is given by

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$

for $0 \le \theta < 2\pi$, $r > 0$ so that the Jacobian is

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r.$$

Since the density of $(X,Y)'$ is

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{ -\frac{1}{2(1-\rho^2)}(x^2 - 2\rho xy + y^2) \right\}, \quad -\infty < x, y < \infty,$$

it now follows from Theorem I.2.1 that the density of $(R,\Theta)'$ is

$$\begin{aligned} f_{R,\Theta}(r,\theta) &= f_{X,Y}(r\cos\theta, r\sin\theta) \cdot |J| \\ &= r f_{X,Y}(r\cos\theta, r\sin\theta) \\ &= \frac{r}{2\pi\sqrt{1-\rho^2}} \exp\left\{ -\frac{1}{2(1-\rho^2)}(r^2\cos^2\theta - 2\rho r^2 \sin\theta\cos\theta + r^2\sin^2\theta) \right\} \\ &= \frac{r}{2\pi\sqrt{1-\rho^2}} \exp\left\{ -\frac{r^2(1-\rho\sin 2\theta)}{2(1-\rho^2)} \right\} \end{aligned}$$

for $0 \le \theta < 2\pi$, $r > 0$.

The marginal density for Θ is therefore given by

$$f_\Theta(\theta) = \int_0^\infty \frac{r}{2\pi\sqrt{1-\rho^2}} \exp\left\{-\frac{r^2(1-\rho\sin 2\theta)}{2(1-\rho^2)}\right\} dr$$
$$= \frac{1}{2\pi\sqrt{1-\rho^2}} \int_0^\infty r\exp\left\{-\frac{r^2(1-\rho\sin 2\theta)}{2(1-\rho^2)}\right\} dr.$$

Making the change of variables

$$u = \frac{r^2(1-\rho\sin 2\theta)}{2(1-\rho^2)} \quad \text{so that} \quad \frac{(1-\rho^2)du}{(1-\rho\sin 2\theta)} = r dr$$

implies that

$$f_\Theta(\theta) = \frac{1}{2\pi\sqrt{1-\rho^2}} \cdot \frac{(1-\rho^2)}{(1-\rho\sin 2\theta)} \int_0^\infty e^{-u} du = \frac{\sqrt{1-\rho^2}}{2\pi(1-\rho\sin 2\theta)}$$

provided $0 \le \theta < 2\pi$.

Exercise 7.1, page 134: By Definition I, we know that X and $Y-\rho X$ are normally distributed. Therefore, by Theorem 7.1, X and $Y-\rho X$ are independent if and only if $\operatorname{cov}(X, Y-\rho X) = 0$. We compute

$$\operatorname{cov}(X, Y-\rho X) = \operatorname{cov}(X,Y) - \operatorname{cov}(X,\rho X) = \operatorname{cov}(X,Y) - \rho\operatorname{var}(X) = \rho\operatorname{SD}(X)\operatorname{SD}(Y) - \rho\operatorname{var}(X)$$
$$= \rho\operatorname{var}(X) - \rho\operatorname{var}(X) = 0$$

since $\operatorname{SD}(X)\cdot\operatorname{SD}(Y) = \operatorname{SD}(X)\cdot\operatorname{SD}(X) = \operatorname{var}(X)$ by the assumption that $\operatorname{var}(X) = \operatorname{var}(Y)$. Hence, X and $Y-\rho X$ are in fact independent.

Problem #3, page 143: If the random vector $(X,Y)'$ has a multivariate normal distribution, then it follows from Definition I that both $X+Y$ and $X-Y$ are normal random variables. If $\operatorname{var}(X) = \operatorname{var}(Y)$, then

$$\operatorname{cov}(X+Y, X-Y) = \operatorname{cov}(X,X) - \operatorname{cov}(X,Y) + \operatorname{cov}(Y,X) + \operatorname{cov}(Y,Y) = \operatorname{var}(X) - \operatorname{var}(Y) = 0.$$

Theorem V.7.1 therefore implies that $X+Y$ and $X-Y$ are independent since $\operatorname{cov}(X+Y, X-Y) = 0$.

Problem #9, page 144: Note that by Theorem 7.1, in order to show X_1, X_2, and X_3 are independent, it is enough to show that $\operatorname{cov}(X_1,X_2) = \operatorname{cov}(X_1,X_3) = \operatorname{cov}(X_2,X_3) = 0$. Thus, if X_1 and X_2+X_3 are independent, then $\operatorname{cov}(X_1, X_2+X_3) = \operatorname{cov}(X_1,X_2) + \operatorname{cov}(X_1,X_3) = 0$ and so

$$\operatorname{cov}(X_1,X_2) = -\operatorname{cov}(X_1,X_3). \tag{1}$$

If X_2 and X_1+X_3 are independent, then $\operatorname{cov}(X_2, X_1+X_3) = \operatorname{cov}(X_2,X_1) + \operatorname{cov}(X_2,X_3) = 0$ and so

$$\operatorname{cov}(X_2,X_1) = -\operatorname{cov}(X_2,X_3). \tag{2}$$

Finally, if X_3 and X_1+X_2 are independent, then $\operatorname{cov}(X_3, X_1+X_2) = \operatorname{cov}(X_3,X_1) + \operatorname{cov}(X_3,X_2) = 0$ and so

$$\operatorname{cov}(X_3,X_1) = -\operatorname{cov}(X_3,X_2). \tag{3}$$

Since (1), (2), and (3) must be simultaneously satisfied, the only possibility is that $\text{cov}(X_1, X_2) = \text{cov}(X_1, X_3) = \text{cov}(X_2, X_3) = 0$. Hence, X_1, X_2, and X_3 are independent as required.

Problem #10, page 145: Using Theorem V.3.1, the distribution of $\mathbf{Y} = (Y_1, Y_2)'$ is

$$\mathbf{Y} \in N\left(\begin{pmatrix} 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 10 & 5 \\ 5 & 5 \end{pmatrix}\right)$$

and so we see that $Y_1 \in N(2, 10)$, $Y_2 \in N(-1, 5)$, and $\text{corr}(Y_1, Y_2) = \frac{1}{\sqrt{2}}$. Thus, by the results in Section V.6, the distribution of $Y_1|Y_2 = y$ is normal with mean $2 + \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{10}}{\sqrt{5}}(y - (-1)) = y + 3$ and variance $10\left(1 - \left(\frac{1}{\sqrt{2}}\right)^2\right) = 5$. That is,

$$Y_1|Y_2 = y \in N(y + 3, 5).$$

Problem #11, page 145: Using Theorem V.3.1, the distribution of $\mathbf{Y} = (Y_1, Y_2)'$ is

$$\mathbf{Y} \in N\left(\begin{pmatrix} 0 \\ 8 \end{pmatrix}, \begin{pmatrix} 16 & -2 \\ -2 & 16 \end{pmatrix}\right)$$

and so we see that $Y_1 \in N(0, 16)$, $Y_2 \in N(8, 16)$, and $\text{corr}(Y_1, Y_2) = -\frac{1}{8}$. Thus, by the results in Section V.6, the distribution of $Y_1|Y_2 = 10$ is normal with mean $0 - \frac{1}{8} \cdot \frac{4}{4}(10 - 8) = -\frac{1}{4}$ and variance $16\left(1 - \left(-\frac{1}{8}\right)^2\right) = \frac{63}{4}$. That is,

$$Y_1|Y_2 = 10 \in N\left(-\tfrac{1}{4}, \tfrac{63}{4}\right).$$

Problem #12, page 145: Let $\mathbf{X} = (X_1, X_2, X_3)'$ where X_1, X_2, X_3 are i.i.d. $N(1, 1)$ so that $\mathbf{X} \in N(\boldsymbol{\mu}, \mathbf{\Lambda})$ where

$$\boldsymbol{\mu} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{\Lambda} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let $\mathbf{Y} = (U, V)'$ where $U = 2X_1 - X_2 + X_3$ and $V = X_1 + 2X_2 + 3X_3$. If

$$B = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 2 & 3 \end{pmatrix}$$

then $\mathbf{Y} = B\mathbf{X}$. By Theorem 3.1, $\mathbf{Y}$ is MVN with mean

$$B\boldsymbol{\mu} = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \end{pmatrix}$$

and covariance matrix

$$B\mathbf{\Lambda}B' = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -1 & 2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 6 & 3 \\ 3 & 14 \end{pmatrix}.$$

We can immediately conclude that $U \in N(2, 6)$, $V \in N(6, 14)$, and $\text{cov}(U, V) = 3$ so that $\text{corr}(U, V) = \frac{3}{\sqrt{6}\sqrt{14}} = \frac{3}{2\sqrt{21}}$. It follows from Section V.6 that the distribution of $V|U = u$ is

$$N\left(6 + \frac{3}{2\sqrt{21}}\frac{\sqrt{14}}{\sqrt{6}}(u - 2), 14\left(1 - \frac{9}{4 \cdot 21}\right)\right).$$

Choosing $u = 3$ therefore implies that

$$V|U = 3 \in N(6.5, 12.5).$$

Problem #13, page 145: Using Theorem V.3.1, the distribution of $\mathbf{X} = (X_1, X_2, X_3)'$ is

$$\mathbf{X} \in N\left(\begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}2 & 4 & -5\\ 4 & 9 & -10\\ -5 & -10 & 13\end{pmatrix}\right)$$

and so we see that $X_1 \in N(0, 2)$, $X_2 \in N(0, 9)$, and $X_3 \in N(0, 13)$. Since $\text{cov}(X_1, X_3) = -5$, we conclude that $X_1 + X_3 \in N(0, 5)$. Finally, we compute $\text{cov}(X_2, X_1 + X_3) = \text{cov}(X_2, X_1) + \text{cov}(X_2, X_3) = 4 - 10 = -6$ and so $\text{corr}(X_2, X_1 + X_3) = -\frac{2}{\sqrt{5}}$. Thus, by the results in Section V.6, the distribution of $X_2|X_1 + X_3 = x$ is normal with mean $0 - \frac{2}{\sqrt{5}} \cdot \frac{3}{\sqrt{5}}(x - 0) = -\frac{6x}{5}$ and variance $9\left(1 - \left(-\frac{2}{\sqrt{5}}\right)^2\right) = \frac{9}{5}$. That is,

$$X_2|X_1 + X_3 = x \in N\left(-\tfrac{6x}{5}, \tfrac{9}{5}\right).$$

Problem #14, page 145: Using Theorem V.3.1, the distribution of $\mathbf{Y} = (Y_1, Y_2, Y_3)'$ is

$$\mathbf{Y} \in N\left(\begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}2 & 1 & 1\\ 1 & 2 & 1\\ 1 & 1 & 2\end{pmatrix}\right).$$

By definition,

$$f_{Y_1|Y_2=0,Y_3=0}(y) = \frac{f_{Y_1,Y_2,Y_3}(y, 0, 0)}{f_{Y_2,Y_3}(0, 0)}.$$

From Definition III, we know

$$f_{Y_1,Y_2,Y_3}(y, 0, 0) = \left(\frac{1}{2\pi}\right)^{3/2} \frac{1}{\sqrt{4}}\, e^{-\frac{1}{2}\frac{3}{4}y^2}$$

since

$$\begin{pmatrix}2 & 1 & 1\\ 1 & 2 & 1\\ 1 & 1 & 2\end{pmatrix}^{-1} = \frac{1}{4}\begin{pmatrix}3 & -1 & -1\\ -1 & 3 & -1\\ -1 & -1 & 3\end{pmatrix}.$$

The joint distribution of $(Y_2, Y_3)'$ is

$$(Y_2, Y_3)' \in N\left(\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}\right)$$

and so

$$f_{Y_2,Y_3}(0, 0) = \frac{1}{2\pi\sqrt{3}}.$$

Thus, we conclude

$$f_{Y_1|Y_2=0,Y_3=0}(y) = \frac{\left(\frac{1}{2\pi}\right)^{3/2} \frac{1}{\sqrt{4}}\, e^{-\frac{1}{2}\frac{3}{4}y^2}}{\frac{1}{2\pi\sqrt{3}}} = \frac{1}{\sqrt{2\pi}}\frac{\sqrt{3}}{2} \exp\left\{-\frac{1}{2}\left(\frac{y}{2/\sqrt{3}}\right)^2\right\}$$

which we recognize as the density function of a normal random variable with mean 0 and variance 3/4. That is,

$$Y_1|Y_2 = Y_3 = 0 \in N\left(0, \tfrac{3}{4}\right).$$