Stat 351 Fall 2008
Assignment #6 Solutions

1. We find

$$P(X < Y < Z) = \iiint\limits_{\{x<y<z\}} f(x,y,z)dx\,dy\,dz.$$

Since $f(x,y,z) = e^{-x-y-z}$ for $0 < x, y, z < \infty$, the six equivalent iterated integrals for this expression are

$$\int_0^\infty \int_0^z \int_0^y e^{-x-y-z}dx\,dy\,dz = \int_0^\infty \int_y^\infty \int_0^y e^{-x-y-z}dx\,dz\,dy = \int_0^\infty \int_0^z \int_x^z e^{-x-y-z}dy\,dx\,dz$$

$$= \int_0^\infty \int_x^\infty \int_x^z e^{-x-y-z}dy\,dz\,dx = \int_0^\infty \int_0^y \int_y^\infty e^{-x-y-z}dz\,dx\,dy = \int_0^\infty \int_x^\infty \int_y^\infty e^{-x-y-z}dz\,dy\,dx.$$

All of them integrate to 1/6; for example, the first of these iterated integrals is

$$\begin{aligned}
\int_0^\infty \int_0^z \int_0^y e^{-x-y-z}dx\,dy\,dz &= \int_0^\infty \int_0^z e^{-y-z} \int_0^y e^{-x}dx\,dy\,dz \\
&= \int_0^\infty \int_0^z e^{-y-z}(1-e^{-y})\,dy\,dz \\
&= \int_0^\infty e^{-z} \int_0^z (e^{-y} - e^{-2y})\,dy\,dz \\
&= \int_0^\infty e^{-z} \left[(1-e^{-z}) - \left(\frac{1}{2} - \frac{1}{2}e^{-2z} \right) \right] dz \\
&= \int_0^\infty \left(\frac{1}{2}e^{-z} - e^{-2z} + \frac{1}{2}e^{-3z} \right) dz \\
&= \frac{1}{2} - \frac{1}{2} + \frac{1}{6}
\end{aligned}$$

as expected.

Problem #3, page 115: If $0 \le y \le 1/2$, then

$$f_Y(y) = \int_y^{1-y} f_{X_{(1)},X_{(2)}}(y,z)\,dz = \int_y^{1-y} 2\,dz = 2(1-2y).$$

On the other hand, if $1/2 \le y \le 1$, then

$$f_Y(y) = \int_{1-y}^{y} f_{X_{(1)},X_{(2)}}(z,1-y)\,dz = \int_{1-y}^{y} 2\,dz = 2(2y-1).$$

Problem #6, page 115: Since $E[F(X_{(n)}) - F(X_{(1)})] = E[F(X_{(n)})] - E[F(X_{(1)})]$, we compute each of $E[F(X_{(n)})]$ and $E[F(X_{(1)})]$ separately. Therefore, by definition,

$$E[F(X_{(n)})] = \int_{-\infty}^{\infty} F(y_n) f_{X_{(n)}}(y_n)\,dy_n.$$

From Theorem IV.1.2, we know that $f_{X_{(n)}}(y_n) = n[F_{X_{(n)}}(y_n)]^{n-1} f(y_n)$ so that

$$\int_{-\infty}^{\infty} F(y_n) f_{X_{(n)}}(y_n)\,dy_n = \int_{-\infty}^{\infty} n[F(y_n)]^n f(y_n)\,dy_n.$$

Making the substitution $u = F(y_n)$ so that $du = F'(y_n)dy_n = f(y_n)dy_n$ gives

$$\int_{-\infty}^{\infty} n[F(y_n)]^n f(y_n)\, dy_n = \int_0^1 nu^n\, du = \frac{n}{n+1}.$$

Note that since F is a distribution, our new limits of integration are $F(-\infty) = 0$ and $F(\infty) = 1$. As for $E[F(X_{(1)})]$, using Theorem IV.1.2, we compute

$$E[F(X_{(1)})] = \int_{-\infty}^{\infty} F(y_1) f_{X_{(1)}}(y_1)\, dy_1 = \int_{-\infty}^{\infty} F(y_1) n[1 - F(y_1)]^{n-1} f(y_1)\, dy_1.$$

Making the same substitution as above gives

$$\int_{-\infty}^{\infty} F(y_1) n[1 - F(y_1)]^{n-1} f(y_1)\, dy_1 = \int_0^1 nu(1-u)^{n-1}\, du = n\int_0^1 (1-v)v^{n-1}\, dv = 1 - \frac{n}{n+1}.$$

Finally, we combine our two results to conclude that

$$E[F(X_{(n)}) - F(X_{(1)})] = \frac{n}{n+1} - \left[1 - \frac{n}{n+1}\right] = \frac{n-1}{n+1}.$$

Problem #9, page 116: **(a)**: If X_1 and X_2 are independent Exp(a) random variables, then by Theorem IV.2.1, the joint density of $(X_{(1)}, X_{(2)})$ is given by

$$f_{X_{(1)},X_{(2)}}(y_1, y_2) = \begin{cases} \frac{2}{a^2} \exp\left(-\frac{y_1+y_2}{a}\right), & \text{for } 0 < y_1 < y_2 < \infty, \\ 0, & \text{otherwise.} \end{cases}$$

Suppose that $U = X_{(1)}$ and let $V = X_{(2)} - X_{(1)}$. Solving for $X_{(1)}$ and $X_{(2)}$ gives

$$X_{(1)} = U \quad \text{and} \quad X_{(2)} = U + V.$$

The Jacobian of this transformation is given by

$$J = \begin{vmatrix} \frac{\partial y_1}{\partial u} & \frac{\partial y_1}{\partial v} \\ \frac{\partial y_2}{\partial u} & \frac{\partial y_2}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1.$$

Therefore, by Theorem I.2.1, the density of (U, V) is given by

$$f_{U,V}(u,v) = f_{X_{(1)},X_{(2)}}(u, u+v)\cdot|J| = \frac{2}{a^2}\exp\left(-\frac{u+u+v}{a}\right) = \frac{2}{a^2}\exp\left(-\frac{2u+v}{a}\right) = \frac{2}{a}e^{-2u/a}\cdot\frac{1}{a}e^{-v/a}$$

provided that $v > 0$ and $u > 0$. The marginal density of U is

$$f_U(u) = \int_{-\infty}^{\infty} f_{U,V}(u,v)\, dv = \int_0^{\infty} \frac{2}{a}e^{-2u/a}\cdot\frac{1}{a}e^{-v/a}\, dv = \frac{2}{a}e^{-2u/a}$$

for $u > 0$. We recognize that this is the density of an exponential random variable with parameter $a/2$; that is, $U = X_{(1)} \in \text{Exp}(a/2)$. The marginal density of V is

$$f_V(v) = \int_{-\infty}^{\infty} f_{U,V}(u,v)\, du = \int_0^{\infty} \frac{2}{a}e^{-2u/a}\cdot\frac{1}{a}e^{-v/a}\, du = \frac{1}{a}e^{-v/a}$$

for $v > 0$. We recognize that this is the density of an exponential random variable with parameter a; that is, $V = X_{(2)} - X_{(1)} \in \text{Exp}(a)$. Since we can express $f_{U,V}(u,v) = f_U(u) \cdot f_V(v)$ we conclude that U and V are independent; in other words, $X_{(1)}$ and $X_{(2)} - X_{(1)}$ are independent.

(b): To compute $E(X_{(2)}|X_{(1)} = y)$, we can use properties of conditional expectation (Theorem II.2.2):

$$\begin{aligned} E(X_{(2)}|X_{(1)} = y) &= E(X_{(2)} - X_{(1)} + X_{(1)}|X_{(1)} = y) \\ &= E(X_{(2)} - X_{(1)}|X_{(1)} = y) + E(X_{(1)}|X_{(1)} = y) \\ &= E(X_{(2)} - X_{(1)}) + y \\ &= a + y \end{aligned}$$

where the first expression after the third equality follows since $X_{(2)} - X_{(1)}$ is independent of $X_{(1)}$ and the second expression follows since $X_{(1)}$ is "known" when conditioned on the value $X_{(1)} = y$.

As for $E(X_{(1)}|X_{(2)} = x)$, we need to compute this by definition of conditional expectation. That is,

$$f_{X_{(1)}|X_{(2)}=x}(y_1) = \frac{f_{X_{(1)},X_{(2)}}(y_1, x)}{f_{X_{(2)}}(x)} = \frac{\frac{2}{a^2}e^{-y_1/a} \cdot e^{-x/a}}{\frac{2}{a}(1 - e^{-x/a}) \cdot e^{-x/a}} = \frac{1}{a} \frac{e^{-y_1/a}}{1 - e^{-x/a}}$$

provided $0 < y_1 < x$. This then gives

$$E(X_{(1)}|X_{(2)} = x) = \int_{-\infty}^{\infty} f_{X_{(1)}|X_{(2)}=x}(y_1)\, dy_1 = \int_0^x \frac{y_1}{a} \frac{e^{-y_1/a}}{1 - e^{-x/a}}\, dy_1 = \frac{1}{a(1 - e^{-x/a})} \int_0^x y_1\, e^{-y_1/a}\, dy_1.$$

Integrating by parts gives

$$\int_0^x y_1\, e^{-y_1/a}\, dy_1 = a^2 - a^2 e^{-x/a} - axe^{-x/a}.$$

Therefore,

$$E(X_{(1)}|X_{(2)} = x) = \frac{a^2 - a^2 e^{-x/a} - axe^{-x/a}}{a(1 - e^{-x/a})} = a - \frac{xe^{-x/a}}{1 - e^{-x/a}} = a - \frac{x}{e^{x/a} - 1}.$$

Problem #10, page 116: Let X_1, X_2, and X_3 are independent, identically distributed $U(0,1)$ random variables. Notice that if $x > 1/2$, then since $X_{(3)} > X_{(1)}$ we conclude

$$P(X_{(3)} > \tfrac{1}{2} \,|\, X_{(1)} = x) = 1.$$

On the other hand, suppose that $0 \le x \le 1/2$. By equation (3.10) on page 114,

$$f_{X_{(1)},X_{(3)}}(y_1, y_3) = 6(y_3 - y_1)$$

provided $0 < y_1 < y_3 < 1$. Therefore, we find

$$P(X_{(3)} > \tfrac{1}{2} \,|\, X_{(1)} = x) = \frac{\displaystyle\int_{1/2}^1 f_{X_{(1)},X_{(3)}}(x, y_3)\, dy_3}{f_{X_{(1)}}(x)}.$$

For the numerator we calculate

$$\int_{1/2}^{1} f_{X_{(1)},X_{(3)}}(x, y_3)\, dy_3 = \int_{1/2}^{1} 6(y_3 - x)\, dy_3 = (3y_3^2 - 6xy_3)\Big|_{1/2}^{1} = \frac{9}{4} - 3x = \frac{3}{4}(3 - 4x).$$

As for the denominator, from Remark 3.1 on page 114, we find

$$f_{X_{(1)}}(x) = 3(1 - x)^2$$

provided $0 < x < 1$. Putting these pieces together, we conclude

$$P(X_{(3)} > \tfrac{1}{2} \,|\, X_{(1)} = x) = \frac{\frac{3}{4}(3 - 4x)}{3(1 - x)^2} = \frac{(3 - 4x)}{4(1 - x)^2}.$$

That is,

$$P(X_{(3)} > \tfrac{1}{2} \,|\, X_{(1)} = x) = \begin{cases} \frac{(3-4x)}{4(1-x)^2}, & \text{if } 0 \le x \le 1/2, \\ 1, & \text{if } x > 1/2. \end{cases}$$

Problem #12, page 116: Since $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ are i.i.d. $U(0, a)$ random variables, we conclude from Theorem IV.1.2 that $X_{(n)}$ and $Y_{(n)}$ are independent and identically distributed $\beta(1, n)$ random variables. In order to simplify matters we let $X = X_{(n)}$ and $Y = Y_{(n)}$ so that X and Y have common density function

$$f(x) = \frac{n}{a^n} x^{n-1}, \quad 0 < x < a$$

and common distribution function

$$F(x) = \begin{cases} 0, & x \le 0, \\ \frac{x^n}{a^n}, & 0 < x < a, \\ 1, & x \ge 1. \end{cases}$$

If we now let $S = \min\{X, Y\}$ and $T = \max\{X, Y\}$, then Theorem IV.2.1 implies that the joint density of (S, T) is

$$f_{S,T}(s, t) = 2 \cdot \frac{n}{a^n} s^{n-1} \cdot \frac{n}{a^n} t^{n-1} = \frac{2n^2}{a^{2n}} s^{n-1} t^{n-1}, \quad 0 < s < t < a.$$

The next step is to let $U = \frac{T}{S}$ and $V = S$ so that $S = V$ and $T = UV$. We find the Jacobian of this transformation is

$$J = \begin{vmatrix} \frac{\partial s}{\partial u} & \frac{\partial s}{\partial v} \\ \frac{\partial t}{\partial u} & \frac{\partial t}{\partial v} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ v & u \end{vmatrix} = -v.$$

The density of (U, V) is therefore given by

$$f_{U,V}(u, v) = f_{S,T}(v, uv) \cdot |J| = \frac{2n^2}{a^{2n}} v^{n-1} (uv)^{n-1} \cdot v = \frac{2n^2}{a^{2n}} u^{n-1} v^{2n-1}$$

provided that $1 < u < \infty$, $0 < v < \frac{a}{u} < a$. The marginal density for U is therefore given by

$$f_U(u) = \int_{-\infty}^{\infty} f_{U,V}(u, v)\, \mathrm{d}v = \frac{2n^2}{a^{2n}} u^{n-1} \int_0^{a/u} v^{2n-1}\, \mathrm{d}v = \frac{n}{a^{2n}} u^{n-1} v^{2n} \Big|_{v=0}^{v=a/u} = \frac{n}{a^{2n}} u^{n-1} \frac{a^{2n}}{u^{2n}} = n u^{-(n+1)}$$

provided that $1 < u < \infty$. Since we are interested in

$$Z_n = n \log\left(\frac{\max\{X_{(n)}, Y_{(n)}\}}{\min\{X_{(n)}, Y_{(n)}\}}\right) = n \log U$$

we can now use techniques from Chapter I to find the density of Z_n. Let $Z = Z_n = n \log U$. Therefore, $F_Z(z) = P(Z \le z) = P(U \le e^{z/n})$ and so

$$f_Z(z) = \frac{1}{n} e^{z/n} f_U(e^{z/n}) = \frac{1}{n} e^{z/n} \cdot n(e^{z/n})^{-(n+1)} = e^{-z}$$

provided that $0 < z < \infty$. Hence we conclude that $Z_n \in \text{Exp}(1)$.

Problem #17, page 117: The key observation is that

$$X_1 + X_2 + \cdots + X_n = X_{(1)} + X_{(2)} + \cdots + X_{(n)}.$$

Using the fact from Stat 251 that the sum of n independent and identically distributed $\text{Exp}(a)$ random variables has a gamma distribution with parameters a and n, we conclude that

$$\sum_{i=1}^{n} X_{(i)} \in \Gamma(a, n).$$

Problem #7, page 115: By Theorem IV.3.1, the joint density

$$f_{X_{(1)},X_{(2)},X_{(3)},X_{(4)}}(y_1, y_2, y_3, y_4) = 24$$

provided that $0 < y_1 < y_2 < y_3 < y_4 < 1$. Therefore,

$$\begin{aligned} f_{X_{(3)},X_{(4)}}(y_3, y_4) = \int_0^{y_3} \int_0^{y_2} f_{X_{(1)},X_{(2)},X_{(3)},X_{(4)}}(y_1, y_2, y_3, y_4)\, dy_1\, dy_2 &= \int_0^{y_3} \int_0^{y_2} 24\, dy_1\, dy_2 \\ &= \int_0^{y_3} 24 y_2\, dy_2 \\ &= 12 y_3^2 \end{aligned}$$

provided that $0 < y_3 < y_4 < 1$. We then see that

$$P(X_{(3)} + X_{(4)} \le 1) = \int_0^{1/2} \int_y^{1-y} f_{X_{(3)},X_{(4)}}(y, z)\, dz\, dy$$

where $f_{X_{(3)},X_{(4)}}(y, z) = 12y^2$ for $0 < y < z < 1$. This then gives

$$P(X_{(3)} + X_{(4)} \le 1) = \int_0^{1/2} \int_y^{1-y} 12y^2 \; dz\, dy = \int_0^{1/2} 12y^2(1-2y)\, dy = (4y^3 - 6y^4)\Big|_0^{1/2} = \frac{1}{8}.$$

Problem #8, page 115: By definition,

$$\rho_{X_{(1)},X_{(3)}} = \frac{\text{cov}(X_{(1)}, X_{(3)})}{\sqrt{\text{var}(X_{(1)}) \cdot \text{var}(X_{(3)})}}.$$

Since X_1, X_2, X_3 are iid $\text{Exp}(1)$ random variables, we conclude from Theorem IV.2.1 that

$$f_{X_{(1)},X_{(3)}}(y_1, y_3) = 6(e^{-y_1} - e^{-y_3})e^{-y_1}e^{-y_3}$$

provided $0 < y_1 < y_3 < \infty$. We also conclude from Theorem IV.1.2 (or, equivalently, page 103) that

$$f_{X_{(1)}}(y_1) = 3(e^{-y_1})^2 e^{-y_1} = 3e^{-3y_1}$$

provided that $0 < y_1 < \infty$, and that

$$f_{X_{(3)}}(y_3) = 3(1 - e^{-y_3})^2 e^{-y_3}$$

provided that $0 < y_3 < \infty$. Since we recognize $X_{(1)} \in \text{Exp}(1/3)$ we conclude immediately that $E(X_{(1)}) = 1/3$ and $\text{var}(X_{(1)}) = 1/9$. Next we compute

$$\begin{aligned} E(X_{(3)}) = \int_0^\infty 3y_3(1 - e^{-y_3})^2 e^{-y_3}\, dy_3 &= \int_0^\infty 3y_3 e^{-y_3}\, dy_3 - \int_0^\infty 6y_3 e^{-2y_3}\, dy_3 + \int_0^\infty 3y_3 e^{-3y_3}\, dy_3 \\ &= 3\Gamma(2) - 6\left(\frac{1}{2}\right)^2 \Gamma(2) + 3\left(\frac{1}{3}\right)^2 \Gamma(2) \\ &= \frac{11}{6} \end{aligned}$$

and

$$\begin{aligned} E(X_{(3)}^2) = \int_0^\infty 3y_3^2(1 - e^{-y_3})^2 e^{-y_3}\, dy_3 &= \int_0^\infty 3y_3^2 e^{-y_3}\, dy_3 - \int_0^\infty 6y_3^2 e^{-2y_3}\, dy_3 + \int_0^\infty 3y_3^2 e^{-3y_3}\, dy_3 \\ &= 3\Gamma(3) - 6\left(\frac{1}{2}\right)^3 \Gamma(3) + 3\left(\frac{1}{3}\right)^3 \Gamma(3) \\ &= \frac{85}{18}. \end{aligned}$$

Therefore,

$$\text{var}(X_{(3)}) = E(X_{(3)}^2) - [E(X_{(3)})]^2 = \frac{85}{18} - \left(\frac{11}{6}\right)^2 = \frac{49}{36}.$$

Now we compute

$$\begin{aligned} E(X_{(1)}X_{(3)}) &= \int_{-\infty}^\infty \int_{-\infty}^\infty f_{X_{(1)},X_{(3)}}(y_1, y_3)\, dy_3\, dy_1 = \int_0^\infty \int_{y_1}^\infty 6y_1y_3(e^{-y_1} - e^{-y_3})e^{-y_1}e^{-y_3}\, dy_3\, dy_1 \\ &= 6\int_0^\infty y_1 e^{-y_1} \int_{y_1}^\infty y_3(e^{-y_1} - e^{-y_3})e^{-y_3}\, dy_3\, dy_1. \\ &= 6\int_0^\infty y_1 e^{-2y_1} \int_{y_1}^\infty y_3 e^{-y_3}\, dy_3\, dy_1 - 6\int_0^\infty y_1 e^{-y_1} \int_{y_1}^\infty y_3 e^{-2y_3}\, dy_3\, dy_1 \\ &= 6\int_0^\infty y_1 e^{-2y_1}(y_1 e^{-y_1} + e^{-y_1})\, dy_1 - 6\int_0^\infty y_1 e^{-y_1}\left(\frac{1}{2}y_1 e^{-2y_1} + \frac{1}{4}e^{-2y_1}\right) dy_1 \\ &= 3\int_0^\infty y_1^2 e^{-3y_1}\, dy_1 + \frac{9}{2}\int_0^\infty y_1 e^{-3y_1}\, dy_1 \\ &= 3\Gamma(3)\left(\frac{1}{3}\right)^3 + \frac{9}{2}\Gamma(2)\left(\frac{1}{3}\right)^2 \\ &= \frac{13}{18} \end{aligned}$$

so that

$$\text{cov}(X_{(1)}, X_{(3)}) = E(X_{(1)}X_{(3)}) - E(X_{(1)})E(X_{(3)}) = \frac{13}{18} - \frac{1}{3} \cdot \frac{11}{6} = \frac{1}{9}.$$

Finally, we put everything together so that

$$\rho_{X_{(1)},X_{(3)}} = \frac{\text{cov}(X_{(1)}, X_{(3)})}{\sqrt{\text{var}(X_{(1)}) \cdot \text{var}(X_{(3)})}} = \frac{1/9}{1/3 \cdot 7/6} = \frac{2}{7}.$$

Problem #13, page 116: **(a)** If $Y_1 = X_{(1)}$ and $Y_k = X_{(k)} - X_{(k-1)}$, $k = 2, \dots, n$, then solving for $X_{(1)}, X_{(2)}, \dots, X_{(n)}$ gives

$$X_{(1)} = Y_1 \quad \text{and} \quad X_{(k)} = Y_1 + \cdots + Y_k,\ k = 2, \dots, n.$$

The Jacobian of this transformation is given by

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \cdots & \frac{\partial x_1}{\partial y_n} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} & \cdots & \frac{\partial x_2}{\partial y_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial y_1} & \frac{\partial x_n}{\partial y_2} & \cdots & \frac{\partial x_n}{\partial y_n} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & \cdots & 1 & 0 \\ 1 & 1 & 1 & 1 & \cdots & 1 & 1 \end{vmatrix} = 1.$$

(Since the matrix is lower triangular, the determinant is simply the product of the diagonal entries.) By Theorem I.2.1, we have

$$f_{Y_1,\dots,Y_n}(y_1, y_2, \dots, y_n) = f_{X_{(1)},\dots,X_{(n)}}(y_1, y_1 + y_2, \dots, y_1 + y_2 + \cdots + y_n).$$

Since $X_1, \dots, X_n$ are Exp(a) random variables so that they have common density $f(x) = \frac{1}{a}e^{-x/a}$, $x > 0$, we find from Theorem IV.3.1 that the joint density of the order statistic is given by

$$f_{X_{(1)},\dots,X_{(n)}}(x_1, \dots, x_n) = n! \prod_{i=1}^{n} f(x_i) = n! \prod_{i=1}^{n} \frac{1}{a} e^{-x_i/a} = \frac{n!}{a^n} \exp\left\{ -\frac{1}{a} \sum_{i=1}^{n} x_i \right\}$$

provided $0 < x_1 < x_2 < \cdots < x_n$. Hence, we conclude

$$\begin{aligned} f_{Y_1,\dots,Y_n}(y_1, y_2, \dots, y_n) &= f_{X_{(1)},\dots,X_{(n)}}(y_1, y_1 + y_2, \dots, y_1 + y_2 + \cdots + y_n \\ &= \frac{n!}{a^n} \exp\left\{ -\frac{1}{a}(ny_1 + (n-1)y_2 + \cdots + 2y_{n-1} + y_n) \right\} \\ &= \frac{n}{a} e^{-ny_1/a} \cdot \frac{(n-1)}{a} e^{-(n-1)y_2/a} \cdots \frac{2}{a} e^{-2y_{n-1}/a} \cdot \frac{1}{a} e^{-y_n/a} \end{aligned}$$

provided that $0 < y_1 < y_1 + y_2 < \cdots < y_1 + \cdots + y_n$, or equivalently, $y_1 > 0$, $y_2 > 0, \dots, y_n > 0$. In other words, the density function of Y_k is

$$f_{Y_k}(y_k) = \frac{(n+1-k)}{a} \exp\left\{ -\frac{(n+1-k)y_k}{a} \right\}, \quad y_k > 0,$$

so that

$$Y_k \in \text{Exp}\left(\frac{a}{n+1-k} \right).$$

(b) Note that

$$Y_1 + Y_2 + \cdots + Y_n = X_{(1)} + \big(X_{(2)} - X_{(1)}\big) + \cdots + \big(X_{(n)} - X_{(n-1)}\big) = X_{(n)}$$

as in **(a)**. Therefore, since $Y_k \in \text{Exp}(\frac{a}{n+1-k})$, we conclude

$$\mathbb{E}(X_{(n)}) = \mathbb{E}(Y_1) + \mathbb{E}(Y_2) + \cdots + \mathbb{E}(Y_n) = \frac{a}{n+1-1} + \frac{a}{n+1-2} + \cdots + \frac{a}{n+1-n} = a\sum_{k=1}^{n} \frac{1}{k}$$

and

$$\text{var}(X_{(k)}) = \text{var}(Y_1) + \text{var}(Y_2) + \cdots + \text{var}(Y_k) = \frac{a^2}{(n+1-1)^2} + \frac{a^2}{(n+1-2)^2} + \cdots + \frac{a^2}{(n+1-n)^2} = a^2\sum_{k=1}^{n} \frac{1}{k^2}.$$

Problem #14, page 116: **(a)** This is identical to Problem #13(a). Hence, $Y_k \in \text{Exp}(\frac{1}{n+1-k})$ for $k = 1, 2, \ldots, n$.

(b) As in Problem #13(b), we see that $Y_1 + Y_2 + \cdots + Y_n = X_{(n)}$ and so

$$\mathbb{E}(X_{(n)}) = \sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

However, we can also find $\mathbb{E}(X_{(n)})$ another way. By Theorem IV.1.2, we know that

$$f_{X_{(n)}}(x) = n(1 - e^{-x})^{n-1} e^{-x}, \quad 0 < x < \infty$$

and so

$$\mathbb{E}(X_{(n)}) = \int_{-\infty}^{\infty} x f_{X_{(n)}}(x)\, \mathrm{d}x = \int_0^{\infty} nx(1 - e^{-x})^{n-1} e^{-x}\, \mathrm{d}x.$$

Equating these two expressions for $\mathbb{E}(X_{(n)})$ gives

$$\int_0^{\infty} nx(1 - e^{-x})^{n-1} e^{-x}\, \mathrm{d}x = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

as required.

Problem #15, page 117: As in Problem #13(b) we see that

$$X_{(k)} = Y_1 + Y_2 + \cdots + Y_k, \quad k = 1, 2, \ldots, n,$$

where $Y_j \in \text{Exp}(\frac{1}{n+1-j})$ with $Y_1, \ldots, Y_n$ independent. This implies that

$$\begin{aligned} Z_n &= nX_{(1)} + (n-1)X_{(2)} + \cdots + 2X_{(n-1)} + X_{(n)} \\ &= nY_1 + (n-1)(Y_1 + Y_2) + \cdots + 2(Y_1 + Y_2 + \cdots + Y_{n-1}) + (Y_1 + Y_2 + \cdots + Y_n) \\ &= Y_1(1 + 2 + \cdots + n) + Y_2(1 + 2 + \cdots + n - 1) + \cdots + Y_{n-1}(1 + 2) + Y_n. \end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbb{E}(Z_n) &= (1+2+\cdots+n)\mathbb{E}(Y_1) + (1+2+\cdots+n-1)\mathbb{E}(Y_2) + \cdots + (1+2)\mathbb{E}(Y_{n-1}) + \mathbb{E}(Y_n) \\
&= (1+2+\cdots+n)\cdot\frac{1}{n+1-1} + (1+2+\cdots+n-1)\cdot\frac{1}{n+1-2} + \cdots + (1+2)\cdot\frac{1}{n+1-(n-1)} + 1 \\
&= \frac{n(n+1)}{2}\cdot\frac{1}{n} + \frac{(n-1)n}{2}\cdot\frac{1}{n-1} + \cdots + \frac{2(3)}{2}\cdot\frac{1}{2} + 1 \\
&= \frac{n+1}{2} + \frac{n}{2} + \cdots + \frac{3}{2} + \frac{2}{2} \\
&= \sum_{k=1}^{n} \frac{k+1}{2} \\
&= \frac{n(n+1)}{4} + \frac{n}{2} \\
&= \frac{n(n+3)}{4}.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\operatorname{var}(Z_n) &= (1+2+\cdots+n)^2 \operatorname{var}(Y_1) + (1+2+\cdots+n-1)^2 \operatorname{var}(Y_2) + \cdots + (1+2)^2 \operatorname{var}(Y_{n-1}) + \operatorname{var}(Y_n) \\
&= \left(\frac{n(n+1)}{2}\cdot\frac{1}{n}\right)^2 + \left(\frac{(n-1)n}{2}\cdot\frac{1}{n-1}\right)^2 + \cdots + \left(\frac{2(3)}{2}\cdot\frac{1}{2}\right)^2 + 1 \\
&= \sum_{k=1}^{n} \frac{(k+1)^2}{4} \\
&= \frac{1}{4}\left[\sum_{k=1}^{n+1} k^2\right] - \frac{1}{4} \\
&= \frac{1}{4}\left[\frac{(n+1)(n+2)(2n+3)}{6}\right] - \frac{1}{4} \\
&= \frac{n(2n^2+9n+13)}{24}.
\end{aligned}$$

Problem #20, page 118: Since $X_1, X_2, \ldots$ are i.i.d. $U(0,1)$ random variables, they have common distribution function

$$F(x) = \begin{cases} 0, & x \leq 0, \\ x, & 0 < x < 1, \\ 1, & x \geq 1. \end{cases}$$

Thus, if we let $X_{(n)} = \max\{X_1, \ldots, X_n\}$, then by Theorem IV.1.1, the distribution function of $X_{(n)}$ is

$$F_{X_{(n)}}(y) = \begin{cases} 0, & y \leq 0, \\ y^n, & 0 < y < 1, \\ 1, & y \geq 1. \end{cases}$$

Now let $V = \max\{X_1, \ldots, X_N\}$ where $N \in \text{Po}(\lambda)$ is independent of $X_1, X_2, \ldots$. If we condition on the value of N, then there are two cases to consider. Either $N = 0$ which happens with probability $P(N=0) = e^{-\lambda}$ and so

$$P(V=0) = P(N=0) = e^{-\lambda},$$

or $N \geq 1$ in which case the distribution function of $V|N=n$, $n=1,2,3,\ldots$, is given by

$$F_{V|N=n}(y) = \begin{cases} 0, & y \leq 0, \\ y^n, & 0 < y < 1, \\ 1, & y \geq 1. \end{cases}$$

Thus, the density function of $V|N=n$, $n=1,2,3,\ldots$, is given by

$$f_{V|N=n}(y) = ny^{n-1}, \quad 0 < y < 1.$$

Finally, we conclude using the law of total probability that the (unconditional) density of V (in the case $N \geq 1$) is

$$\begin{aligned} f_V(y) = \sum_{n=1}^{\infty} f_{V|N=n}(y) P(N=n) = \sum_{n=1}^{\infty} ny^{n-1} \cdot \frac{\lambda^n e^{-\lambda}}{n!} &= \frac{e^{-\lambda}}{y} \sum_{n=1}^{\infty} \frac{(\lambda y)^n}{(n-1)!} \\ &= \frac{e^{-\lambda}}{y} \cdot \lambda y \sum_{n=1}^{\infty} \frac{(\lambda y)^{n-1}}{(n-1)!} \\ &= \lambda e^{-\lambda} e^{\lambda y} \\ &= \lambda e^{-\lambda(1-y)} \end{aligned}$$

To summarize, we have

- $P(V=0) = e^{-\lambda}$, and
- $f_V(v) = \lambda e^{-\lambda(1-v)}$, $0 < v < 1$.

Notice that

$$P(V=0) + \int_0^1 f_V(v)dv = e^{-\lambda} + \int_0^1 \lambda e^{-\lambda(1-v)} dv = e^{-\lambda} + e^{-\lambda}(e^{\lambda} - 1) = 1$$

as expected. Note that V is an example of a random variable which is neither continuous nor discrete. The expected value of V is given by

$$\begin{aligned} \mathbb{E}(V) = 0 \cdot P(V=0) + \int_0^1 v f_V(v) dv = \int_0^1 \lambda v e^{-\lambda(1-v)} dv &= e^{-\lambda} \int_0^1 \lambda v e^{\lambda v} dv \\ &= \frac{e^{-\lambda}}{\lambda} \int_0^{\lambda} u e^u du \\ &= \frac{e^{-\lambda}}{\lambda} \left[u e^u - e^u \right] \Big|_{u=0}^{u=\lambda} \\ &= \frac{e^{-\lambda}}{\lambda} \left(\lambda e^{\lambda} - e^{\lambda} + 1 \right) \\ &= \frac{\lambda - 1 + e^{-\lambda}}{\lambda} \\ &= 1 - \frac{1}{\lambda} + \frac{e^{-\lambda}}{\lambda}. \end{aligned}$$