Stat 351 Fall 2008
Solutions to Assignment #5

1. (a) We begin by calculating $\mathbb{E}(Y_1)$. That is,

$$\mathbb{E}(Y_1) = 1 \cdot P(Y = 1) + (-1) \cdot P(Y = -1) = p - (1 - p) = 2p - 1.$$

We now notice that $S_{n+1} = S_n + Y_{n+1}$. Therefore,

$$\begin{aligned}
\mathbb{E}(S_{n+1}|X_1, \ldots, X_n) &= \mathbb{E}(S_n + Y_{n+1}|X_1, \ldots, X_n) \\
&= \mathbb{E}(S_n|X_1, \ldots, X_n) + \mathbb{E}(Y_{n+1}|X_1, \ldots, X_n) \\
&= S_n + \mathbb{E}(Y_{n+1}) \\
&= S_n + 2p - 1
\end{aligned}$$

and so

$$\begin{aligned}
\mathbb{E}(X_{n+1}|X_1, \ldots, X_n) &= \mathbb{E}(S_{n+1} - (n + 1)(2p - 1)|X_1, \ldots, X_n) \\
&= \mathbb{E}(S_{n+1}|X_1, \ldots, X_n) - (n + 1)(2p - 1) \\
&= S_n + 2p - 1 - (n + 1)(2p - 1) \\
&= S_n - n(2p - 1) \\
&= X_n.
\end{aligned}$$

We now conclude that $\{X_n, n = 1, 2, \ldots\}$ is, in fact, a martingale.

1. (b) Notice that we can write X_{n+1} as

$$\begin{aligned}
X_{n+1} = S_{n+1} - (n + 1)(2p - 1) &= S_n + Y_{n+1} - n(2p - 1) - (2p - 1) \\
&= X_n + Y_{n+1} - (2p - 1)
\end{aligned}$$

and so

$$\begin{aligned}
X_{n+1}^2 &= (X_n + Y_{n+1})^2 + (2p - 1)^2 - 2(2p - 1)(X_n + Y_{n+1}) \\
&= X_n^2 + Y_{n+1}^2 + 2X_nY_{n+1} + (2p - 1)^2 - 2(2p - 1)(X_n + Y_{n+1}).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\mathbb{E}(X_{n+1}^2|X_1, \ldots, X_n) \\
&\quad = \mathbb{E}(X_n^2|X_1, \ldots, X_n) + \mathbb{E}(Y_{n+1}^2|X_1, \ldots, X_n) + 2\mathbb{E}(X_nY_{n+1}|X_1, \ldots, X_n) \\
&\qquad + (2p - 1)^2 - 2(2p - 1)\mathbb{E}(X_n + Y_{n+1}|X_1, \ldots, X_n) \\
&\quad = X_n^2 + \mathbb{E}(Y_{n+1})^2 + 2X_n\mathbb{E}(Y_{n+1}) + (2p - 1)^2 - 2(2p - 1)(X_n + \mathbb{E}(Y_{n+1})) \\
&\quad = X_n^2 + 1 + 2(2p - 1)X_n + (2p - 1)^2 - 2(2p - 1)(X_n + (2p - 1)) \\
&\quad = X_n^2 + 1 + 2(2p - 1)X_n + (2p - 1)^2 - 2(2p - 1)X_n - 2(2p - 1)^2 \\
&\quad = X_n^2 + 1 - (2p - 1)^2,
\end{aligned}$$

and so we find

$$\begin{aligned}
\mathbb{E}(Z_{n+1}|X_1,\ldots,X_n) &= \mathbb{E}(X_{n+1}^2|X_1,\ldots,X_n) - 4(n+1)p(1-p) \\
&= X_n^2 + 1 - (2p-1)^2 - 4(n+1)p(1-p) \\
&= X_n^2 + 1 - (4p^2 - 4p + 1) - 4np(1-p) - 4p(1-p) \\
&= X_n^2 + 1 - 4p^2 + 4p - 1 - 4np(1-p) - 4p + 4p^2 \\
&= X_n^2 - 4np(1-p) \\
&= Z_n.
\end{aligned}$$

Hence, $\{Z_n, n = 1, 2, \ldots\}$ is, in fact, a martingale.

2. We interpret the conditional probabilities given in the problem to mean

$$P(X_{n+1} = (1-q)x_n|X_n = x_n) = 1 - x_n \quad \text{and} \quad P(X_{n+1} = q + (1-q)x_n|X_n = x_n) = x_n.$$

Therefore,

$$\begin{aligned}
&\mathbb{E}(X_{n+1}|X_1 = x_1,\ldots,X_n = x_n) \\
&\quad = (1-q)x_n \cdot P(X_{n+1} = (1-q)x_n|X_n = x_n) + (q + (1-q)x_n) \cdot P(X_{n+1} = q + (1-q)x_n|X_n = x_n) \\
&\quad = (1-q)x_n \cdot (1 - x_n) + (q + (1-q)x_n) \cdot x_n \\
&\quad = (1-q)x_n - (1-q)x_n^2 + qx_n + (1-q)x_n^2 \\
&\quad = x_n - qx_n + qx_n \\
&\quad = x_n.
\end{aligned}$$

In other words,

$$\mathbb{E}(X_{n+1}|X_1,\ldots,X_n) = X_n$$

and so we conclude that $\{X_n, n = 1, 2, \ldots\}$ is, in fact, a martingale.