Stat 351 Fall 2008
Assignment #3 Solutions

Problem #3, page 27: Suppose that $T \in t(n)$ so that the density of T is given by

$$f_T(x) = \frac{\Gamma(\frac{n+1}{2})}{\sqrt{\pi n}\,\Gamma(\frac{n}{2})} \cdot \left(1 + \frac{x^2}{n}\right)^{-(n+1)/2}, \quad -\infty < x < \infty.$$

Let $Y = T^2$. If $y \geq 0$, then the distribution function of Y is given by

$$\begin{aligned} F_Y(y) = P(Y \leq y) = P(T^2 \leq y) = P(-\sqrt{y} \leq T \leq \sqrt{y}) &= \int_{-\sqrt{y}}^{\sqrt{y}} f_T(x)\,dx \\ &= \int_0^{\sqrt{y}} f_T(x)\,dx - \int_0^{-\sqrt{y}} f_T(x)\,dx. \end{aligned}$$

Taking derivatives with respect to y gives

$$\begin{aligned} f_Y(y) = f_T(\sqrt{y}) \cdot \frac{1}{2\sqrt{y}} - f_T(-\sqrt{y}) \cdot \frac{-1}{2\sqrt{y}} &= \frac{1}{2\sqrt{y}}\,(\,f_T(\sqrt{y}) + f_T(-\sqrt{y})\,) \\ &= \frac{\Gamma(\frac{n+1}{2})}{\sqrt{\pi n y}\,\Gamma(\frac{n}{2})} \cdot \left(1 + \frac{y}{n}\right)^{-(n+1)/2} \\ &= \frac{\Gamma(\frac{1+n}{2})\left(\frac{1}{n}\right)^{1/2}}{\Gamma(\frac{1}{2})\,\Gamma(\frac{n}{2})} \cdot \frac{y^{1/2-1}}{\left(1+\frac{y}{n}\right)^{(1+n)/2}}, \quad y \geq 0. \end{aligned}$$

In order to write this last line, we have used the fact that $\Gamma(1/2) = \sqrt{\pi}$. Notice that this is the density of an $F(1,n)$ random variable. (See page 261.)

Problem #5, page 27: Suppose that $X \in C(0,1)$ so that the density of X is given by

$$f_X(x) = \frac{1}{\pi} \cdot \frac{1}{1+x^2}, \quad -\infty < x < \infty.$$

Let $Y = X^2$. If $y \geq 0$, then the distribution function of Y is given by

$$\begin{aligned} F_Y(y) = P(Y \leq y) = P(X^2 \leq y) = P(-\sqrt{y} \leq X \leq \sqrt{y}) &= \int_{-\sqrt{y}}^{\sqrt{y}} f_X(x)\,dx \\ &= \int_0^{\sqrt{y}} f_X(x)\,dx - \int_0^{-\sqrt{y}} f_X(x)\,dx. \end{aligned}$$

Taking derivatives with respect to y gives

$$f_Y(y) = f_X(\sqrt{y}) \cdot \frac{1}{2\sqrt{y}} - f_X(-\sqrt{y}) \cdot \frac{-1}{2\sqrt{y}} = \frac{1}{2\sqrt{y}}\,(\,f_X(\sqrt{y}) + f_X(-\sqrt{y})\,) = \frac{1}{\pi\sqrt{y}} \cdot \frac{1}{1+y}, \quad y \geq 0.$$

Notice that this is the density of an $F(1,1)$ random variable. (See page 261 and recall that $\Gamma(1) = 1$, $\Gamma(1/2) = \sqrt{\pi}$.)

Problem #6, page 27: If $X \in \beta(1,1)$, then the density function of X is

$$f_X(x) = \frac{\Gamma(1+1)}{\Gamma(1)\Gamma(1)} x^{1-1}(1-x)^{1-1} = 1, \quad 0 < x < 1.$$

(We have used the fact that $\Gamma(2) = \Gamma(1) = 1$.) Since the density of X is also that of a uniform random variable, we conclude $X \in U(0,1)$. Therefore, $\beta(1,1) = U(0,1)$.

Problem #9, page 27: Suppose that $X \in N(0,1)$ and $Y \in \chi^2(n)$ are independent random variables. Let $U = \frac{X}{\sqrt{Y/n}}$ and $V = \sqrt{Y/n}$ so that solving for X and Y gives

$$X = UV \quad \text{and} \quad Y = nV^2.$$

The Jacobian of this transformation is given by

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ 0 & 2nv \end{vmatrix} = 2nv^2.$$

The density of (U,V) is therefore given by

$$f_{U,V}(u,v) = f_{X,Y}(uv, nv^2) \cdot |J| = 2nv^2 f_X(uv) f_Y(nv^2)$$

using the assumed independence of X and Y. Substituting in the corresponding densities gives

$$f_{U,V}(u,v) = 2nv^2 \frac{1}{\sqrt{2\pi}} e^{-u^2v^2/2} \frac{1}{\Gamma(n/2)} (nv^2)^{n/2-1} 2^{-n/2} e^{-nv^2/2} = \frac{n^{n/2}}{2^{n/2-1/2}\sqrt{\pi}\,\Gamma(n/2)} v^n e^{-v^2(u^2+n)/2}$$

provided that $-\infty < u < \infty$, $0 < v < \infty$. The marginal density of U is

$$f_U(u) = \int_{-\infty}^{\infty} f_{U,V}(u,v)\, dv = \frac{n^{n/2}}{2^{n/2-1/2}\sqrt{\pi}\,\Gamma(n/2)} \int_0^{\infty} v^n e^{-v^2(u^2+n)/2}\, dv.$$

Making the substitution $z = v^2(u^2+n)/2$ so that $dz = v(u^2+n)dv$ gives

$$\begin{aligned}
\frac{n^{n/2}}{2^{n/2-1/2}\sqrt{\pi}\,\Gamma(n/2)} \int_0^{\infty} v^n e^{-v^2(u^2+n)/2}\, dv &= \frac{n^{n/2}}{2^{n/2-1/2}\sqrt{\pi}\,\Gamma(n/2)} (u^2+n)^{-1/2-n/2} 2^{n/2-1/2} \int_0^{\infty} z^{n/2-1/2} e^{-z}\, dz \\
&= \frac{n^{n/2}}{\sqrt{\pi}\,\Gamma(n/2)} (u^2+n)^{-1/2-n/2} \Gamma(n/2+1/2) \\
&= \frac{n^{n/2}\Gamma(n/2+1/2)}{\sqrt{\pi}\,\Gamma(n/2)} (u^2+n)^{-1/2-n/2} \\
&= \frac{\Gamma(\frac{n+1}{2})}{\sqrt{n\pi}\,\Gamma(\frac{n}{2})} \frac{1}{(1+\frac{u^2}{n})^{(n+1)/2}}
\end{aligned}$$

provided $-\infty < u < \infty$. We recognize that this is the density of a $t(n)$ random variable (see page 261), and so we conclude that $U = \frac{X}{\sqrt{Y/n}} \in t(n)$.

Problem #10, page 28: Suppose that $X \in \chi^2(m)$ and $Y \in \chi^2(n)$ are independent random variables. Let $U = \frac{X/m}{Y/n}$ and $V = Y/n$ so that solving for X and Y gives

$$X = mUV \quad \text{and} \quad Y = nV.$$

The Jacobian of this transformation is given by

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} mv & mu \\ 0 & n \end{vmatrix} = mnv.$$

The density of (U, V) is therefore given by

$$f_{U,V}(u,v) = f_{X,Y}(muv, nv) \cdot |J| = mnv f_X(muv) f_Y(nv)$$

using the assumed independence of X and Y. Substituting in the corresponding densities gives

$$\begin{aligned} f_{U,V}(u,v) &= mnv \frac{1}{\Gamma(m/2)} (muv)^{m/2-1} 2^{-m/2} e^{-muv/2} \frac{1}{\Gamma(n/2)} (nv)^{n/2-1} 2^{-n/2} e^{-nv/2} \\ &= \frac{m^{m/2} n^{n/2} 2^{-m/2-n/2}}{\Gamma(m/2)\Gamma(n/2)} v^{m/2+n/2-1} u^{m/2-1} e^{-v(mu+n)/2} \end{aligned}$$

provided that $0 < u < \infty$, $0 < v < \infty$. The marginal density of U is

$$f_U(u) = \int_{-\infty}^{\infty} f_{U,V}(u,v)\, dv = \frac{m^{m/2} n^{n/2} 2^{-m/2-n/2}}{\Gamma(m/2)\Gamma(n/2)} u^{m/2-1} \int_0^{\infty} v^{m/2+n/2-1} e^{-v(mu+n)/2}\, dv.$$

Making the substitution $z = v(mu+n)/2$ so that $dz = (mu+n)/2 dv$ gives

$$\begin{aligned} &\frac{m^{m/2} n^{n/2} 2^{-m/2-n/2}}{\Gamma(m/2)\Gamma(n/2)} u^{m/2-1} \int_0^{\infty} v^{m/2+n/2-1} e^{-v(mu+n)/2}\, dv \\ &\qquad = \frac{m^{m/2} n^{n/2} 2^{-m/2-n/2}}{\Gamma(m/2)\Gamma(n/2)} u^{m/2-1} 2^{m/2+n/2} (mu+n)^{-m/2-n/2} \int_0^{\infty} z^{m/2+n/2+1} e^{-z}\, dz \\ &\qquad = \frac{m^{m/2} n^{n/2}}{\Gamma(m/2)\Gamma(n/2)} u^{m/2-1} (mu+n)^{-m/2-n/2} \Gamma(m/2+n/2) \\ &\qquad = \frac{\Gamma(\frac{m+n}{2})}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})} m^{m/2} n^{n/2} \frac{u^{m/2-1}}{(mu+n)^{(m+n)/2}} \\ &\qquad = \frac{\Gamma(\frac{m+n}{2}) \left(\frac{m}{n}\right)^{n/2}}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})} \frac{u^{m/2-1}}{(1+\frac{mu}{n})^{(m+n)/2}} \end{aligned}$$

provided $0 < u < \infty$. We recognize that this is the density of a $F(m,n)$ random variable (see page 261), and so we conclude that $U = \frac{X/m}{Y/n} \in F(m,n)$.

Problem #11, page 28: If $X \in \text{Exp}(a)$, then a quick calculation shows that $\frac{2X}{a} \in \text{Exp}(2)$. However, comparing the exponential and chi-square densities (see page 260), we see that $\text{Exp}(2) = \chi^2(2)$. Similarly, $2Y/a \in \text{Exp}(2) = \chi^2(2)$. Thus, using the result of Problem #10, we conclude that

$$\frac{X}{Y} = \frac{2X/a}{2Y/a} \in F(2,2).$$

Problem #23, page 29: Suppose that X and Y have joint density

$$f_{X,Y}(x,y) = \begin{cases} \frac{x}{(1+x)^2(1+xy)^2}, & \text{for } x, y > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Let $U = X$ and $V = XY$ so that solving for X and Y gives

$$X = U \quad \text{and} \quad Y = V/U.$$

The Jacobian of this transformation is given by

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -v/u^2 & 1/u \end{vmatrix} = \frac{1}{u}.$$

The density of (U, V) is therefore given by

$$f_{U,V}(u, v) = f_{X,Y}(u, v/u) \cdot |J| = \frac{u}{(1+u)^2(1+u \cdot v/u)^2} \cdot \frac{1}{u} = \frac{1}{(1+u)^2} \cdot \frac{1}{(1+v)^2},$$

provided that $0 < u < \infty$, $0 < v < \infty$. Since we can write the joint density as a product of a function of u only multiplied by a function of v only, we conclude that U and V are independent. That is,

$$f_{U,V}(u, v) = f_U(u) \cdot f_V(v)$$

where

$$f_U(u) = \frac{1}{(1+u)^2} \text{ for } u > 0, \quad \text{and} \quad f_V(v) = \frac{1}{(1+v)^2} \text{ for } v > 0.$$

Notice that both U and V have the same distribution, namely $F(2, 2)$. (See page 261.)

Problem #24, page 29: Suppose that X and Y have joint density

$$f_{X,Y}(x, y) = \begin{cases} \frac{2}{(1+x+y)^3} & \text{for } x, y > 0, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Let $U = X + Y$ and $V = \frac{X}{X+Y}$ so that solving for X and Y gives

$$X = UV \quad \text{and} \quad Y = U - UV.$$

The Jacobian of this transformation is given by

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ 1-v & -u \end{vmatrix} = -u.$$

The density of (U, V) is therefore given by

$$f_{U,V}(u, v) = f_{X,Y}(uv, u - uv) \cdot |J| = \frac{2}{(1 + uv + u - uv)^3} \cdot u = \frac{2u}{(1+u)^3},$$

provided that $0 < u < \infty$, $0 < v < 1$. Since we can write the joint density as a product of a function of u only multiplied by a function of v only, we conclude that

$$f_{U,V}(u, v) = f_U(u) \cdot f_V(v)$$

where

$$f_U(u) = \frac{2u}{(1+u)^3} \text{ for } u > 0, \quad \text{and} \quad f_V(v) = 1 \text{ for } 0 < v < 1.$$

Therefore, the density of $X + Y$ is

$$f_{X+Y}(u) = \frac{2u}{(1+u)^3} \text{ for } u > 0.$$

(b) Let $U = X - Y$ and $V = X$, so that solving for X and Y gives

$$X = V \quad \text{and} \quad Y = V - U.$$

The Jacobian of this transformation is given by

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ -1 & 1 \end{vmatrix} = 1.$$

The density of (U, V) is therefore given by

$$f_{U,V}(u,v) = f_{X,Y}(v, v-u) \cdot |J| = \frac{2}{(1+v+v-u)^3} \cdot 1 = \frac{2}{(1+2v-u)^3},$$

provided that $v > u$ and $v > 0$ (i.e., $v > \max\{0, u\}$), and $-\infty < u < \infty$. If $u > 0$, then $\max\{u, 0\} = u$, and so we calculate

$$f_U(u) = \int_{-\infty}^{\infty} f_{U,V}(u,v)\,dv = \int_u^{\infty} \frac{2}{(1+2v-u)^3}\,dv = \frac{1}{2(1+2v-u)^2}\bigg|_u^{\infty} = \frac{1}{2(1+u)^2}.$$

If $u \le 0$, then $\max\{u, 0\} = 0$, and so we calculate

$$f_U(u) = \int_{-\infty}^{\infty} f_{U,V}(u,v)\,dv = \int_0^{\infty} \frac{2}{(1+2v-u)^3}\,dv = \frac{1}{2(1+2v-u)^2}\bigg|_0^{\infty} = \frac{1}{2(1-u)^2}.$$

Therefore, the density of $X - Y$ is

$$f_{X-Y}(u) = \frac{1}{2(1+|u|)^2} \quad \text{for } -\infty < u < \infty.$$

Problem #25, page 30: Suppose that $U = X^2 Y$ and let $V = X$. Solving for X and Y gives

$$X = V \quad \text{and} \quad Y = \frac{U}{V^2}.$$

The Jacobian of this transformation is given by

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ v^{-2} & -2uv^{-3} \end{vmatrix} = -v^{-2}.$$

If the density of (X, Y) is

$$f_{X,Y}(x,y) = \begin{cases} e^{-x^2 y}, & \text{for } x \ge 1,\ y > 0, \\ 0, & \text{otherwise,} \end{cases}$$

then the density of (U, V) is therefore given by

$$f_{U,V}(u,v) = f_{X,Y}(v, uv^{-2}) \cdot |J| = \frac{1}{v^2}\,e^{-u}$$

provided that $v \ge 1$ and $u > 0$. We can now determine the density of U as follows.

Routine Way: The marginal density of U is

$$f_U(u) = \int_{-\infty}^{\infty} f_{U,V}(u,v)\,dv = \int_1^{\infty} \frac{1}{v^2}\,e^{-u}\,dv = e^{-u}\left[-v^{-1}\right]_1^{\infty} = e^{-u}$$

for $u > 0$. We recognize that this is the density of an exponential random variable with parameter 1; that is, $U = X^2Y \in \text{Exp}(1)$.

Slick Way: Since the joint density of (U, V) is

$$f_{U,V}(u,v) = \begin{cases} v^{-2}e^{-u}, & \text{for } v \geq 1,\ u > 0, \\ 0, & \text{otherwise,} \end{cases}$$

we can immediately conclude that U and V are independent random variables with $f_V(v) = v^{-2}$ for $v \geq 1$ and $f_U(u) = e^{-u}$ for $u > 0$. And so we find (as before) that $U = X^2Y \in \text{Exp}(1)$.

Problem #26, page 30: Suppose that X and Y have joint density

$$f_{X,Y}(x,y) = \begin{cases} \lambda^2 e^{-\lambda y}, & \text{for } 0 < x < y, \\ 0, & \text{otherwise.} \end{cases}$$

Let $U = Y$ and $V = \frac{X}{Y-X}$ so that solving for X and Y gives

$$X = \frac{UV}{1+V} \quad \text{and} \quad Y = U.$$

The Jacobian of this transformation is given by

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v(1+v)^{-1} & u(1+v)^{-2} \\ 1 & 0 \end{vmatrix} = -\frac{u}{(1+v)^2}.$$

The density of (U, V) is therefore given by

$$f_{U,V}(u,v) = f_{X,Y}(uv(1+v)^{-1}, u) \cdot |J| = \lambda^2 e^{-\lambda u} \cdot \frac{u}{(1+v)^2} = \lambda^2 u\, e^{-\lambda u} \cdot \frac{1}{(1+v)^2},$$

provided that $0 < u < \infty$, $0 < v < \infty$. Since we can write the joint density as a product of a function of u only multiplied by a function of v only, we conclude that U and V are independent. That is,

$$f_{U,V}(u,v) = f_U(u) \cdot f_V(v)$$

where

$$f_U(u) = \lambda^2 u\, e^{-\lambda u} \text{ for } u > 0, \quad \text{and} \quad f_V(v) = \frac{1}{(1+v)^2} \text{ for } v > 0.$$

Notice that $U \in \Gamma(2, \lambda^{-1})$ and that $V \in F(2, 2)$. (See pages 260–261.)

Problem #27, page 30: Suppose that $X_1 \in \Gamma(a_1, b)$ and $X_2 \in \Gamma(a_2, b)$ are independent random variables so that their joint density is

$$f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y) = \begin{cases} \frac{1}{\Gamma(a_1)\Gamma(a_2)} x_1^{a_1-1} x_2^{a_2-1} \frac{1}{b^{a_1+a_2}} e^{-x_1/b - x_2/b}, & \text{for } x_1 > 0,\ x_2 > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Let $U = \frac{X_1}{X_2}$ and $V = X_1 + X_2$ so that solving for X_1 and X_2 gives

$$X_1 = \frac{UV}{U+1} \quad \text{and} \quad X_2 = \frac{V}{U+1}.$$

The Jacobian of this transformation is given by

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial u} & \frac{\partial x_1}{\partial v} \\ \frac{\partial x_2}{\partial u} & \frac{\partial x_2}{\partial v} \end{vmatrix} = \begin{vmatrix} v(1+u)^{-2} & u(1+u)^{-1} \\ -v(1+u)^{-2} & (1+u)^{-1} \end{vmatrix} = \frac{v}{(1+u)^2}.$$

The density of (U, V) is therefore given by

$$\begin{aligned} f_{U,V}(u,v) &= f_{X_1,X_2}(uv(1+u)^{-1}, v(1+u)^{-1}) \cdot |J| \\ &= \frac{1}{\Gamma(a_1)\Gamma(a_2)}(uv(1+u)^{-1})^{a_1-1}(v(1+u)^{-1})^{a_2-1}\frac{1}{b^{a_1+a_2}}e^{-uv(1+u)^{-1}/b - v(1+u)^{-1}/b} \cdot \frac{v}{(1+u)^2} \\ &= \frac{1}{\Gamma(a_1)\Gamma(a_2)}\frac{1}{b^{a_1+a_2}}\frac{u^{a_1-1}v^{a_1+a_2-1}}{(1+u)^{a_1+a_2}}e^{-v/b} \end{aligned}$$

provided that $0 < u < \infty$, $0 < v < \infty$. The marginal density of U is

$$f_U(u) = \int_{-\infty}^{\infty} f_{U,V}(u,v)\,dv = \frac{1}{\Gamma(a_1)\Gamma(a_2)}\frac{1}{b^{a_1+a_2}}\frac{u^{a_1-1}}{(1+u)^{a_1+a_2}}\int_0^{\infty} v^{a_1+a_2-1}e^{-v/b}\,dv$$

To evaluate

$$\int_0^{\infty} v^{a_1+a_2-1}e^{-v/b}\,dv$$

we make the substitution $z = \frac{v}{b}$ so that $dz = \frac{1}{b}dv$. This implies that

$$\begin{aligned} \int_0^{\infty} v^{a_1+a_2-1}e^{-v/b}\,dv = \int_0^{\infty} (bz)^{a_1+a_2-1}e^{-z}b\,dz &= b^{a_1+a_2}\int_0^{\infty} z^{a_1+a_2-1}e^{-z}\,dz \\ &= b^{a_1+a_2}\Gamma(a_1+a_2). \end{aligned}$$

This now implies that

$$f_U(u) = \frac{1}{\Gamma(a_1)\Gamma(a_2)}\frac{1}{b^{a_1+a_2}}\frac{u^{a_1-1}}{(1+u)^{a_1+a_2}}b^{a_1+a_2}\Gamma(a_1+a_2) = \frac{\Gamma(a_1+a_2)}{\Gamma(a_1)\Gamma(a_2)}\frac{u^{a_1-1}}{(1+u)^{a_1+a_2}}, \quad u > 0,$$

which we recognize as the density of a $\beta(a_1, a_2)$ random variable. (See page 260.) To find the marginal density of V we observe that since we can write the joint density as a product of a function of u only multiplied by a function of v only, we conclude that U and V are independent. That is,

$$f_{U,V}(u,v) = f_U(u) \cdot f_V(v)$$

where

$$f_U(u) = \frac{\Gamma(a_1+a_2)}{\Gamma(a_1)\Gamma(a_2)}\frac{u^{a_1-1}}{(1+u)^{a_1+a_2}}, \quad u > 0,$$

and

$$f_V(v) = \frac{f_{U,V}(u,v)}{f_U(u)} = \frac{\frac{1}{\Gamma(a_1)\Gamma(a_2)}\frac{1}{b^{a_1+a_2}}\frac{u^{a_1-1}v^{a_1+a_2-1}}{(1+u)^{a_1+a_2}}e^{-v/b}}{\frac{\Gamma(a_1+a_2)}{\Gamma(a_1)\Gamma(a_2)}\frac{u^{a_1-1}}{(1+u)^{a_1+a_2}}} = \frac{1}{\Gamma(a_1+a_2)}v^{a_1+a_2-1}\frac{1}{b^{a_1+a_2}}e^{-v/b}, \quad v > 0.$$

Notice that $V = X_1 + X_2 \in \Gamma(a_1 + a_2, b)$. (See page 260.) Alternatively, we can calculate

$$f_V(v) = \int_{-\infty}^{\infty} f_{U,V}(u,v)\,du = \frac{1}{\Gamma(a_1)\Gamma(a_2)}\frac{1}{b^{a_1+a_2}}v^{a_1+a_2-1}e^{-v/b}\int_0^{\infty}\frac{u^{a_1-1}}{(1+u)^{a_1+a_2}}\,du.$$

Observe that

$$\frac{\Gamma(a_1+a_2)}{\Gamma(a_1)\Gamma(a_2)}\frac{u^{a_1-1}}{(1+u)^{a_1+a_2}}, \quad u > 0,$$

is the density of a $\beta(a_1, a_2)$ random variable. This implies that

$$\int_0^{\infty}\frac{u^{a_1-1}}{(1+u)^{a_1+a_2}}\,du = \frac{\Gamma(a_1)\Gamma(a_2)}{\Gamma(a_1+a_2)},$$

and so we conclude that

$$f_V(v) = \frac{1}{\Gamma(a_1+a_2)}\frac{1}{b^{a_1+a_2}}v^{a_1+a_2-1}e^{-v/b}, \quad v > 0,$$

as before.