Stat 351 Fall 2008
Assignment #1 Solutions

**2. (a)** If $X \sim \text{Unif}[1,3]$, then $F_X(x) = \frac{x-1}{2}$ for $1 \le x \le 3$, and if $Y \sim \mathcal{N}(0,1)$, then

$$F_Y(y) = \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du$$

for $-\infty < y < \infty$. Since $X$ and $Y$ are independent, we conclude that

$$F_{X,Y}(x,y) = F_X(x) \cdot F_Y(y) = \frac{x-1}{2} \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du$$

for $1 \le x \le 3$ and $-\infty < y < \infty$. We should also note that if $x < 1$, then $F_X(x) = 0$ and if $x \ge 3$, then $F_X(x) = 1$. Combining everything we conclude

$$F_{X,Y}(x,y) = \begin{cases} \frac{x-1}{2} \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du, & \text{if } 0 \le x \le 2 \text{ and } -\infty < y < \infty, \\ \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du, & \text{if } x > 3 \text{ and } -\infty < y < \infty, \\ 0, & \text{if } x < 1 \text{ and } -\infty < y < \infty. \end{cases}$$

**(b)** We find

$$\frac{\partial^2}{\partial x \partial y} F_{X,Y}(x,y) = \frac{\partial^2}{\partial x \partial y} \left[ \frac{x-1}{2} \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du, \right] = \frac{1}{2} \cdot \frac{1}{\sqrt{2\pi}} e^{-y^2/2}.$$

Since $f_X(x) = \frac{1}{2}$, $1 \le x \le 3$, and $f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}$, $-\infty < y < \infty$, we see that

$$\frac{\partial^2}{\partial x \partial y} F_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)$$

as required.

**(c)** If $Z \in \text{Exp}(4)$ is independent of $X$ and $Y$, then the joint density of $(X, Y, Z)$ is given by

$$f_{X,Y,Z}(x,y,z) = f_X(x) \cdot f_Y(y) \cdot f_Z(z) = \frac{1}{2} \cdot \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \cdot \frac{1}{4} e^{-z/4} = \frac{1}{\sqrt{128\pi}} \, e^{-\frac{1}{4}(z+2y^2)}$$

for $1 \le x \le 3$, $-\infty < y < \infty$, and $z > 0$.

**3.** If $X$ and $Y$ are both discrete random variables, and their joint mass function is $p_{X,Y}(x,y)$, then

$$F_{X,Y}(x,y) = \sum_{x' \le x} \sum_{y' \le y} p_{X,Y}(x', y').$$

If $X$ and $Y$ are both continuous random variables, and their joint density function is $f_{X,Y}(x,y)$, then

$$F_{X,Y}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(u,v) \, dv \, du.$$

**4. (a)** Observe that $\text{cov}(X,Z) = \mathbb{E}(XZ) - \mathbb{E}(X)\mathbb{E}(Z) = \mathbb{E}(XZ)$ since $\mathbb{E}(X) = 0$. But $\mathbb{E}(XZ) = \mathbb{E}(X \cdot YX) = \mathbb{E}(X^2 Y) = \mathbb{E}(X^2)\mathbb{E}(Y) = 0$ using the assumed independence of $Y$ and $X$. Hence, we conclude that $\text{cov}(X,Z) = 0$.

**(b)** We see that

$$\begin{aligned} P\{Z \geq 1\} = P\{XY \geq 1\} &= P\{X \geq 1, Y = 1\} + P\{X \leq -1, Y = -1\} \\ &= P\{X \geq 1\}P\{Y = 1\} + P\{X \leq -1\}P\{Y = -1\} \\ &= \frac{1}{2}P\{X \geq 1\} + \frac{1}{2}P\{X \leq -1\} \\ &= P\{X \geq 1\} \end{aligned}$$

using the symmetry of the normal distribution. Since

$$P\{X \geq 1, Z \geq 1\} = P\{X \geq 1, XY \geq 1\} = P\{X \geq 1, Y = 1\} = \frac{1}{2}P\{X \geq 1\}$$

and since

$$P\{Z \geq 1\} \in (0, 1/2),$$

we conclude that

$$P\{X \geq 1, Z \geq 1\} \neq P\{X \geq 1\}P\{Z \geq 1\}$$

which implies that $X$ and $Z$ are not independent. (Note that $P\{X \geq 1\} = P\{Z \geq 1\} \approx 0.1587$.)

**(c)** As in **(b)** we have

$$\begin{aligned} P\{Z \geq x\} = P\{XY \geq x\} &= P\{X \geq x, Y = 1\} + P\{X \leq -x, Y = -1\} \\ &= P\{X \geq x\}P\{Y = 1\} + P\{X \leq -x\}P\{Y = -1\} \\ &= \frac{1}{2}P\{X \geq x\} + \frac{1}{2}P\{X \leq -x\} \\ &= P\{X \geq x\} \end{aligned}$$

using the symmetry of the normal distribution.

Since $P\{X \geq x\} = P\{Z \geq x\}$ is equivalent to saying $P\{X \leq x\} = P\{Z \leq x\}$ which in turn is equivalent to saying that $F_X(x) = F_Z(x)$, we conclude that $X$ and $Z$ have the same distribution (i.e., $Z \in \mathcal{N}(0, 1)$).