Stat 351 Fall 2008
Assignment #9

This assignment is due on Thursday, December 4, 2008, at 3:30 pm in my office. You must submit solutions to all problems. As indicated on the course outline, solutions will be graded for both content and clarity of exposition. The solutions that you submit must be neat and orderly. Do not crowd your work or write in multiple columns. Your assignment must be stapled and problem numbers clearly labelled.

1. Suppose that the random vector $\mathbf{X} = (X_1, X_2)'$ has the multivariate normal distribution

$$\mathbf{X} \in N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right)$$

where $\rho = \text{cov}(X_1, X_2) > 0$.

(a) Prove that there exists a standard normal random variable $Z \in N(0,1)$ such that

$$X_1 = \rho X_2 + \sqrt{1-\rho^2} Z.$$

(b) Prove that Z is independent of X_2.

2.

- Exercise 5.3, page 129
- Problem #27, page 147

3. Suppose that the random vector $\mathbf{X} = (X_1, X_2)' \in \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Lambda})$ with

$$\boldsymbol{\mu} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{\Lambda} = \begin{bmatrix} \frac{7}{4} & -\frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{4} & \frac{5}{4} \end{bmatrix}.$$

(This random vector was considered in Problem #2 on Midterm #2.) Determine the equation of the level curves of the density function $f_{\mathbf{X}}(x_1, x_2)$. Normalize your quadratic form so that it is in standard form, i.e., so that all arbitrary constants equal 1. Express your answer analytically, geometrically, and descriptively.

4.

(a) If $X \in U(0,1)$, show that $-\log X$ has an exponential distribution. (What is the parameter of this exponential distribution?)

(b) Determine the density function of $\prod_{i=1}^{n} X_i$ where $X_1, \ldots, X_n$ are iid $U(0,1)$ random variables.

5. Suppose that $Y_1, Y_2, \ldots$ are i.i.d. with $P(Y_1 = 1) = P(Y_1 = -1) = \frac{1}{2}$. Let $S_0 = 0$ and for $n = 1, 2, 3, \ldots$ set

$$S_n = \sum_{j=1}^{n} Y_j$$

so that $\{S_n, n = 0, 1, 2, \ldots\}$ is a simple random walk.

(a) Compute $\text{cov}(S_n, S_{n+1})$.

(b) For $n = 1, 2, 3, \ldots$, determine an expression for

$$P(S_{2n} = 2x)$$

where $x = -n, -n+1, \ldots, n-1, n$. (Note that the indexing/notation is such that $2n$ and $2x$ necessarily have the same parity.)

6.

Suppose that the number of calls per hour arriving at an answering service follows a Poisson process with $\lambda = 4$.

(a) What is the probability that fewer than two calls come in the first hour?

(b) Suppose that six calls arrive in the first hour. What is the probability that at least two calls will arrive in the second hour?

(c) The person answering the phones waits until fifteen calls have arrived before going on a break. What is the expected amount of time that the person will wait?

(d) Suppose that it is known that exactly eight calls arrived in the first two hours. What is the probability that exactly five of them arrived in the first hour?