Stat 351 Fall 2008
Assignment #7

This assignment is due at the beginning of class on Wednesday, November 5, 2008. You must submit solutions to all problems. As indicated on the course outline, solutions will be graded for both content and clarity of exposition. The solutions that you submit must be neat and orderly. Do not crowd your work or write in multiple columns. Your assignment must be stapled and problem numbers clearly labelled.

1. Suppose that X_1 and X_2 are independent $N(0,1)$ random variables. Set $Y_1 = X_1 + 3X_2 - 2$ and $Y_2 = X_1 - 2X_2 + 1$.

(a) Determine the distributions of Y_1 and Y_2.

(b) Determine the distribution of $\mathbf{Y} = (Y_1, Y_2)'$.

2. Let $\mathbf{X}$ have a three-dimensional normal distribution with mean vector $\boldsymbol{\mu}$ and covariance matrix Λ given by

$$\boldsymbol{\mu} = \begin{pmatrix} 3 \\ 2 \\ -3 \end{pmatrix} \quad \text{and} \quad \Lambda = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 4 & -2 \\ 3 & -2 & 8 \end{pmatrix},$$

respectively. If $Y_1 = X_1 - X_3$ and $Y_2 = 3X_2$, determine the distribution of $\mathbf{Y} = (Y_1, Y_2)'$.

3. Suppose that Y_1, Y_2, and Y_3 are independent $N(0,1)$ random variables. Set

$$\begin{aligned} X_1 &= Y_1 + Y_3, \\ X_2 &= 2Y_1 - Y_2 + 2Y_3, \\ X_3 &= 2Y_1 - 3Y_3. \end{aligned}$$

Determine the distribution of $\mathbf{X} = (X_1, X_2, X_3)'$.

4. Suppose that X_1, X_2, and X_3 are independent $N(0,1)$ random variables. Set

$$\begin{aligned} Y_1 &= X_2 - X_3, \\ Y_2 &= X_1 + 2X_3, \\ Y_3 &= X_1 - 2X_2. \end{aligned}$$

Determine the distribution of $\mathbf{Y} = (Y_1, Y_2, Y_3)'$.

(continued)

5. Suppose that $X \in U(0,1)$, and that the distribution of Y conditioned on $X = x$ is $N(x, x^2)$; that is, $Y|X = x \in N(x, x^2)$.

(a) Find $E(Y)$, $\text{var}(Y)$, and $\text{cov}(X, Y)$. *Hint: Conditional expectations will simplify calculations.*

(b) Prove that Y/X and X are independent.

(c) Use your results of **(a)** and **(b)** to show that

$$E\left(\frac{Y}{X}\right) = \frac{E(Y)}{E(X)}.$$

Note that in general $E(Y/X) \neq E(Y)/E(X)$.

6. Suppose that

$$\mathbf{X} = (X, Y)' \in N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right).$$

Show that the correlation of X^2 and Y^2 is ρ^2.

Some Hints: (i) If $Z \in N(0,1)$, use the moment generating function to calculate $E(Z^4)$.

(ii) In order to calculate the higher moment involving X and Y, using conditional expectations will greatly simplify the calculation. Determine the distribution of $Y|X$. (Use Equation (6.2) on page 130.) Then use Theorem III.2.2 on page 37.