Statistics 351 Fall 2007 Midterm #2 – Solutions

1. (a) Let

$$B = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad \bar{b} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}$$

so that

$$B\mathbf{X} + \bar{b} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} + \begin{pmatrix} -2 \\ 0 \end{pmatrix} = \begin{pmatrix} X_1 - X_2 - 2 \\ X_1 + X_2 \end{pmatrix} = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \mathbf{Y}.$$

By Theorem V.3.1, we conclude that $\mathbf{Y} \in N(B\boldsymbol{\mu} + \bar{b}, B\boldsymbol{\Lambda}B')$ where

$$\mathbb{E}(\mathbf{Y}) = B\boldsymbol{\mu} + \bar{b} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} -2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and

$$\operatorname{cov}(\mathbf{Y}) = B\boldsymbol{\Lambda}B' = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 9 \end{pmatrix}.$$

1. (b) From **(a)**, we can read that $Y_1 \in N(0,1)$, $Y_2 \in N(0,9)$, and $\operatorname{cov}(Y_1, Y_2) = -1$. Hence, $\rho = \operatorname{corr}(Y_1, Y_2) = -1/3$ so that the density of $\mathbf{Y}$ is given by

$$f_{Y_1,Y_2}(y_1, y_2) = \frac{1}{2\pi\sqrt{8}} \exp\left\{ -\frac{1}{2} \left(\frac{9y_1^2}{8} + \frac{2y_1y_2}{8} + \frac{y_2^2}{8} \right) \right\}.$$

1. (c) We now find

$$f_{Y_2|Y_1=0}(y_2) = \frac{f_{\mathbf{Y}}(0, y_2)}{f_{Y_1}(0)} = \frac{\frac{1}{2\pi\sqrt{8}} \exp\left\{ -\frac{1}{2} \left(\frac{9(0)^2}{8} + \frac{2(0)y_2}{8} + \frac{y_2^2}{8} \right) \right\}.}{\frac{1}{\sqrt{2\pi}} \exp\left\{ -\frac{1}{2}(0) \right\}}$$
$$= \frac{1}{\sqrt{8}\sqrt{2\pi}} \exp\left\{ -\frac{1}{16} y_2^2 \right\}$$

In other words, the distribution of $Y_2 | Y_1 = 0$ is $N(0, 8)$.

2. If $\mathbf{Y} = (Y_1, Y_2)'$, then since Y_1 and Y_2 are linear combinations of the components of $\mathbf{X}$, we conclude from Definition I that $\mathbf{Y}$ has a multivariate normal distribution. Therefore, we know from Theorem V.7.1 that Y_1 and Y_2 are independent if and only if $\operatorname{cov}(Y_1, Y_2) = 0$. Since

$$\begin{aligned}
\operatorname{cov}(Y_1, Y_2) &= \operatorname{cov}(2X_1 + X_2 + 1, 3X_1 - 2X_2 - 2) \\
&= \operatorname{cov}(2X_1 + X_2, 3X_1 - 2X_2) \\
&= \operatorname{cov}(2X_1, 3X_1) + \operatorname{cov}(X_2, -2X_2) + \operatorname{cov}(2X_1, -2X_2) + \operatorname{cov}(X_2, 3X_1) \\
&= 6\operatorname{var}(X_1) - 2\operatorname{var}(X_2) - \operatorname{cov}(X_1, X_2) \\
&= 6(1) - 2(4) - \alpha \\
&= -2 - \alpha,
\end{aligned}$$

we see that $\operatorname{cov}(Y_1, Y_2) = 0$ iff $\alpha = -2$.

3. Recall that the necessary conditions for a matrix to be the covariance matrix of some random vector are that it be symmetric and non-negative definite. (This is the content of Definition V.2.1 and Theorem V.2.1.) Hence, we see that A cannot be a covariance matrix since $\det(A) = -8$ so that A is not non-negative definite, D cannot be a covariance matrix since D_2, the upper left 2×2 block of D, has $\det(D_2) = -1$ implying that D is not non-negative definite, and E cannot be a covariance matrix since it is not symmetric.

4. Since we are told that $\mathbf{Z}$ has a multivariate normal distribution, we know from Definition I that Z_1 and Z_2 are one-dimensional normal random variables. Since $\mathbf{X}$ and $\mathbf{Y}$ are independent, we know that $\operatorname{cov}(X_i, Y_j) = 0$ for $i = 1, 2$, $j = 1, 2$. We therefore calculate

- $\mathbb{E}(Z_1) = \mathbb{E}(X_1) + \mathbb{E}(Y_1) = 0 + 1 = 1$,
- $\mathbb{E}(Z_2) = \mathbb{E}(X_2) - \mathbb{E}(Y_2) = 0 - 1 = -1$,
- $\operatorname{var}(Z_1) = \operatorname{var}(X_1) + \operatorname{var}(Y_1) + 2\operatorname{cov}(X_1, Y_1) = 1 + 2 = 3$,
- $\operatorname{var}(Z_2) = \operatorname{var}(X_2) + \operatorname{var}(Y_2) - 2\operatorname{cov}(X_2, Y_2) = 2 + 3 = 5$.

Furthermore, we calculate

$$\begin{aligned}\operatorname{cov}(Z_1, Z_2) = \operatorname{cov}(X_1 + Y_1, X_2 - Y_2) &= \operatorname{cov}(X_1, X_2) - \operatorname{cov}(Y_1, Y_2) + \operatorname{cov}(Y_1, X_2) - \operatorname{cov}(X_1, Y_2) \\ &= \operatorname{cov}(X_1, X_2) - \operatorname{cov}(Y_1, Y_2) \\ &= -1 + 2 = 1.\end{aligned}$$

Hence, we conclude that

$$\boldsymbol{\mu} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{and} \quad \boldsymbol{\Lambda} = \begin{pmatrix} 3 & 1 \\ 1 & 5 \end{pmatrix}.$$

5. Notice that $P(X_{(1)} = X_1, X_{(2)} = X_2, X_{(3)} = X_3) = P(X_1 < X_2 < X_3)$. Therefore, conditioning on the value of X_2 and using the law of total probability gives

$$\begin{aligned}P(X_1 < X_2 < X_3) &= \int_0^\infty P(X_1 < x, X_3 > x | X_2 = x) f_{X_2}(x)\, dx \\ &= \int_0^\infty P(X_1 < x) P(X_3 > x) f_{X_2}(x)\, dx\end{aligned}$$

where the second equality follows from the fact that X_1, X_2, X_3 are independent. Since

$$P(X_1 < x) = \int_0^x e^{-x_1} dx_1 = 1 - e^{-x} \quad \text{and} \quad P(X_3 > x) = \int_x^\infty e^{-x_3} dx_3 = e^{-x},$$

we see that

$$P(X_1 < X_2 < X_3) = \int_0^x (1 - e^{-x}) e^{-x} e^{-x} dx = \int_0^\infty e^{-2x} - e^{-3x} dx = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$