Statistics 351 (Fall 2007)
The Jacobian for Polar Coordinates

Example. Determine the Jacobian for the change-of-variables from cartesian coordinates to polar coordinates.

Solution. The traditional letters to use are

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta.$$

However, to agree with the notation from class, we let

$$x_1 = y_1 \cos y_2 \quad \text{and} \quad x_2 = y_1 \sin y_2.$$

In other words, our original variables are $\overline{x} = (x_1, x_2)$, and our new variables are $\overline{y} = (y_1, y_2)$. The variables $\overline{x}$ and $\overline{y}$ are related through the function $g : \mathbb{R}^2 \to \mathbb{R}^2$ defined implicitly by

$$g(\overline{x}) = g(x_1, x_2) = (g_1(x_1, x_2), g_2(x_1, x_2)) = (y_1, y_2).$$

In other words,

$$y_1 = \sqrt{x_1^2 + x_2^2} \quad \text{and} \quad y_2 = \arctan(x_2/x_1).$$

We now compute the required partial derivatives:

$$\frac{\partial x_1}{\partial y_1} = \cos y_2 \ , \ \frac{\partial x_1}{\partial y_2} = -y_1 \sin y_2 \ , \ \frac{\partial x_2}{\partial y_1} = \sin y_2 \ , \ \frac{\partial x_2}{\partial y_2} = y_1 \cos y_2.$$

Therefore, the Jacobian is given by

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix} = \begin{vmatrix} \cos y_2 & -y_1 \sin y_2 \\ \sin y_2 & y_1 \cos y_2 \end{vmatrix} = y_1 \cos^2 y_2 + y_1 \sin^2 y_2 = y_1.$$

Thus, using the traditional notation, $J = r$. i.e., $\mathrm{d}x\,\mathrm{d}y = r\,\mathrm{d}r\,\mathrm{d}\theta$.