Statistics 351 Fall 2007
Independence of $\overline{X}$ and S^2

Theorem. *Suppose that $X_1, \ldots, X_n$ are independent $\mathcal{N}(0,1)$ random variables. If*

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \quad \text{and} \quad S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$$

denote the sample mean *and* sample variance*, respectively, then $\overline{X}$ and S^2 are independent.*

Our proof of this theorem will follow the outline proposed by Exercise V.8.2.

Proof. Since $X_1, \ldots, X_n$ are i.i.d. $\mathcal{N}(0,1)$, we conclude (using, say moment generating functions) that $\overline{X} \in \mathcal{N}(0, 1/n)$. Similarly, we can show that

$$\begin{aligned}\overline{X} - X_j = \frac{1}{n}(X_1 + \cdots + X_{j-1} + X_{j+1} + \cdots + X_n) - \frac{n-1}{n}X_j &\in \mathcal{N}\left(0, \frac{n-1}{n^2} + \frac{(n-1)^2}{n^2}\right) \\ &= \mathcal{N}\left(0, \frac{n-1}{n}\right)\end{aligned}$$

and so

$$X_j - \overline{X} \in \mathcal{N}\left(0, \frac{n-1}{n}\right)$$

as well. We also note that

$$\operatorname{Cov}(X_j, \overline{X}) = \operatorname{Cov}\left(X_j, \frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n}\sum_{i=1}^{n}\operatorname{Cov}(X_j, X_i) = \frac{1}{n}\operatorname{Cov}(X_j, X_j) = \frac{1}{n},$$

and so for $j \neq k$ it follows that

$$\begin{aligned}\operatorname{Cov}(X_j - \overline{X}, X_k - \overline{X}) &= \operatorname{Cov}(X_j, X_k) - \operatorname{Cov}(X_j, \overline{X}) - \operatorname{Cov}(\overline{X}, X_k) + \operatorname{Cov}(\overline{X}, \overline{X}) \\ &= 0 - \frac{1}{n} - \frac{1}{n} + \frac{1}{n} \\ &= -\frac{1}{n}\end{aligned}$$

using the fact that $\operatorname{Cov}(\overline{X}, \overline{X}) = \operatorname{Var}(\overline{X}) = 1/n$. Similarly,

$$\begin{aligned}\operatorname{Cov}(\overline{X}, X_j - \overline{X}) &= \operatorname{Cov}(X_j, \overline{X}) - \operatorname{Cov}(\overline{X}, \overline{X}) \\ &= \frac{1}{n} - \frac{n}{n^2} \\ &= 0.\end{aligned}$$

Thus, we see that $(\overline{X}, X_1 - \overline{X}, \ldots, X_n - \overline{X})' \in \mathcal{N}(\overline{0}, \mathbf{\Lambda})$ where

$$\mathbf{\Lambda} = \begin{bmatrix} 1/n & 0 & 0 & \cdots & 0 \\ 0 & (n-1)/n & -1/n & \cdots & -1/n \\ 0 & -1/n & (n-1)/n & \cdots & -1/n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -1/n & -1/n & \cdots & (n-1)/n \end{bmatrix}.$$

By a theorem from last lecture (Theorem V.7.2), we conclude from the form of $\mathbf{\Lambda}$ that $\overline{X}$ and $(X_1 - \overline{X}, \ldots, X_n - \overline{X})'$ are independent normal random vectors. It now follows from the transformation theorem (Theorem I.2.1 on page 23) that since $\mathbf{X} = (X_1 - \overline{X}, \ldots, X_n - \overline{X})'$ and $\overline{X}$ are independent, so too are

$$\overline{X} \text{ and } \mathbf{X}'\mathbf{X} = \sum_{i=1}^{n} (X_i - \overline{X})^2.$$

This now implies that $\overline{X}$ and S^2 are independent. $\square$