Statistics 351 (Fall 2007)
Covariance Zero Does Not Imply Independence

The following exercise illustrates that two random variables can have covariance zero yet need not be independent.

Exercise. Consider the random variable X defined by $P(X = -1) = 1/4$, $P(X = 0) = 1/2$, $P(X = 1) = 1/4$. Let the random variable Y be defined as $Y := X^2$. Hence, $P(Y = 0|X = 0) = 1$, $P(Y = 1|X = -1) = 1$, $P(Y = 1|X = 1) = 1$.

- Show that the density of Y is $P(Y = 0) = 1/2$, $P(Y = 1) = 1/2$.
- Find the joint density of (X, Y), and show that X and Y are not independent.
- Find the density of XY, compute $\mathbb{E}(XY)$, and show that X and Y are uncorrelated.

Solution.

- We find the density of Y simply using the *law of total probability*:

$$\begin{aligned} &P(Y = 0) \\ &\quad = P(Y = 0|X = 1)P(X = 1) + P(Y = 0|X = 0)P(X = 0) + P(Y = 0|X = -1)P(X = -1) \\ &\quad = 0 \cdot 1/4 + 1 \cdot 1/2 + 0 \cdot 1/4 \\ &\quad = 1/2, \end{aligned}$$

$$\begin{aligned} &P(Y = 1) \\ &\quad = P(Y = 1|X = 1)P(X = 1) + P(Y = 1|X = 0)P(X = 0) + P(Y = 1|X = -1)P(X = -1) \\ &\quad = 1 \cdot 1/4 + 0 \cdot 1/2 + 1 \cdot 1/4 \\ &\quad = 1/2. \end{aligned}$$

- The joint density of (X, Y) is given by

$$\begin{aligned} P(X = 0, Y = 0) &= P(Y = 0|X = 0)P(X = 0) = 1 \cdot 1/2 = 1/2; \\ P(X = 0, Y = 1) &= P(Y = 1|X = 0)P(X = 0) = 0 \cdot 1/2 = 0; \\ P(X = 1, Y = 0) &= P(Y = 0|X = 1)P(X = 1) = 0 \cdot 1/4 = 0; \\ P(X = 1, Y = 1) &= P(Y = 1|X = 1)P(X = 1) = 1 \cdot 1/4 = 1/4; \\ P(X = -1, Y = 0) &= P(Y = 0|X = -1)P(X = -1) = 0 \cdot 1/4 = 0; \\ P(X = -1, Y = 1) &= P(Y = 1|X = -1)P(X = -1) = 1 \cdot 1/4 = 1/4. \end{aligned}$$

  Since, for example, $P(X = 0, Y = 0) = 1/2$, but $P(X = 0)P(Y = 0) = 1/2 \cdot 1/2 = 1/4$, we see that X and Y cannot be independent.

- The possible values of XY are 0, 1, -1. Hence,

$$P(XY = 0) = P(X = 0, Y = 0) = 1/2$$

  and

$$P(XY = 1) = P(X = 1, Y = 1) = 1/4$$

using the computations above. By the law of total probability,

$$P(XY = -1) = 1/4.$$

(Equivalently, $P(XY = -1) = P(X = -1, Y = 1) = 1/4$.) Thus,

$$\mathbb{E}(XY) = 0 \cdot P(XY = 0) + 1 \cdot P(XY = 1) + (-1) \cdot P(XY = -1) = 0 + 1/4 - 1/4 = 0.$$

Since $\mathbb{E}(X) = 0$ and $\mathbb{E}(Y) = 0$, we see that

$$\mathrm{Cov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) = 0 - 0 = 0;$$

whence X and Y are uncorrelated.