Statistics 351 (Fall 2007)
The Density and Characteristic Function Definitions of Multivariate Normality

Suppose that the random vector $\mathbf{X} = (X, Y)'$ has a multivariate normal distribution with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Lambda}$ given by

$$\boldsymbol{\mu} = \begin{pmatrix} \mu_x \\ \mu_y \end{pmatrix} \quad \text{and} \quad \boldsymbol{\Lambda} = \begin{pmatrix} \sigma_x^2 & \rho\sigma_x\sigma_y \\ \rho\sigma_x\sigma_x & \sigma_y^2 \end{pmatrix}.$$

Note that $\rho = \operatorname{corr}(X, Y)$ in this notation.

The characteristic function of $\mathbf{X}$ is

$$\varphi_{\mathbf{X}}(\mathbf{t}) = \exp\left\{ i\mu_x t_1 + i\mu_y t_2 - \frac{1}{2}\left(\sigma_x^2 t_1^2 + 2\rho\sigma_x\sigma_y t_1 t_2 + \sigma_y^2 t_2^2\right)\right\}$$

which written in matrix notation is

$$\varphi_{\mathbf{X}}(\mathbf{t}) = \exp\left\{ i\mathbf{t}'\boldsymbol{\mu} - \frac{1}{2}\mathbf{t}'\boldsymbol{\Lambda}\mathbf{t}\right\}.$$

The density function of $\mathbf{X}$ is

$$f_{\mathbf{X}}(x, y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left\{ -\frac{1}{2(1-\rho^2)}\left(\left(\frac{x-\mu_x}{\sigma_x}\right)^2 - 2\rho\frac{(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y} + \left(\frac{y-\mu_y}{\sigma_y}\right)^2\right)\right\}$$

which written in matrix notation is

$$f_{\mathbf{X}}(x, y) = \frac{1}{2\pi}\frac{1}{\sqrt{\det\boldsymbol{\Lambda}}} \exp\left\{ -\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})'\boldsymbol{\Lambda}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right\}.$$

Of course, there are some noticeable similarities between these two functions. In particular, if $\boldsymbol{\mu} = (0, 0)'$, then

$$\varphi_{\mathbf{X}}(\mathbf{t}) = \exp\left\{ -\frac{1}{2}\left(\sigma_x^2 t_1^2 + 2\rho\sigma_x\sigma_y t_1 t_2 + \sigma_y^2 t_2^2\right)\right\} = \exp\left\{ -\frac{1}{2}\mathbf{t}'\boldsymbol{\Lambda}\mathbf{t}\right\}$$

and

$$\begin{aligned} f_{\mathbf{X}}(x, y) &= \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left\{ -\frac{1}{2(1-\rho^2)}\left(\left(\frac{x}{\sigma_x}\right)^2 - 2\rho\frac{xy}{\sigma_x\sigma_y} + \left(\frac{y}{\sigma_y}\right)^2\right)\right\} \\ &= \frac{1}{2\pi}\frac{1}{\sqrt{\det\boldsymbol{\Lambda}}} \exp\left\{ -\frac{1}{2}\mathbf{x}'\boldsymbol{\Lambda}^{-1}\mathbf{x}\right\}. \end{aligned}$$

Example: **(a)** Let $\mathbf{X} = (X, Y)'$ have characteristic function

$$\varphi_{\mathbf{X}}(x, y) = \exp\left\{ -\frac{1}{2}(x^2 - 2xy + 2y^2)\right\}.$$

Determine the distribution of $\mathbf{X}$.

(b) Let $\mathbf{X} = (X, Y)'$ have density function

$$f_{\mathbf{X}}(x, y) = \frac{1}{2\pi} \exp\left\{-\frac{1}{2}(x^2 - 2xy + 2y^2)\right\}.$$

Determine the distribution of $\mathbf{X}$.

Solution: **(a)** In this first example, we have used the dummy variables x and y instead of t_1 and t_2 just to emphasize the subtle differences between the characteristic function and the density function. Instead, let's write

$$\varphi_{\mathbf{X}}(t_1, t_2) = \exp\left\{-\frac{1}{2}(t_1^2 - 2t_1t_2 + 2t_2^2)\right\}.$$

We can easily see that $\mathbf{X} = (X, Y)'$ is multivariate normal with mean vector $\boldsymbol{\mu} = (0, 0)'$ and covariance matrix $\mathbf{\Lambda}$ where

$$\mathbf{\Lambda} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}.$$

That is, it is easy to read off the covariance matrix from the characteristic function. Note that $\sigma_x^2 = 1$, $\sigma_y^2 = 2$, and $\rho = -\frac{1}{\sqrt{2}}$.

(b) In the case of the multivariate normal density, it is a little harder to read off the covariance matrix $\mathbf{\Lambda}$. However, we can read off $\mathbf{\Lambda}^{-1}$ with ease! If

$$f_{\mathbf{X}}(x, y) = \frac{1}{2\pi} \exp\left\{-\frac{1}{2}(x^2 - 2xy + 2y^2)\right\}$$

then we see that

$$\mathbf{\Lambda}^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

which implies that

$$\mathbf{\Lambda} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

Note that in this example $\sigma_x^2 = 2$, $\sigma_y^2 = 1$, and $\rho = \frac{1}{\sqrt{2}}$.