Statistics 351 (Fall 2007)
Fisher's F-distribution and Statistical Hypothesis Testing

For Problem #4 on page 27 you are asked to manipulate Fisher's F-distribution. The F-distribution arises in the following context. (See Section 10.9 in the Stat 251 textbook by Wackerly, et al.)

Let $Y_1, \ldots, Y_{n_1}$ be i.i.d. $\mathcal{N}(\mu_1, \sigma_1^2)$, and let $X_1, \ldots, X_{n_2}$ be i.i.d. $\mathcal{N}(\mu_2, \sigma_2^2)$ with both μ_i and σ_i^2 unknown.

Suppose that we wish to test

$$H_0 : \sigma_1^2 = \sigma_2^2 \quad \text{vs.} \quad H_A : \sigma_1^2 > \sigma_2^2.$$

The likelihood ratio test gives the rejection region as

$$RR = \left\{ \frac{S_1^2}{S_2^2} > k \right\}$$

where k is a suitably chosen constant,

$$S_1^2 = \frac{1}{n_1 - 1} \sum_{j=1}^{n_1} (Y_j - \overline{Y})^2 \quad \text{with} \quad \overline{Y} = \frac{1}{n_1} \sum_{j=1}^{n_1} Y_j,$$

and

$$S_2^2 = \frac{1}{n_2 - 1} \sum_{j=1}^{n_2} (X_j - \overline{X})^2 \quad \text{with} \quad \overline{X} = \frac{1}{n_2} \sum_{j=1}^{n_2} X_j.$$

Fact. If

$$Z_1 = \frac{(n_1 - 1)S_1^2}{\sigma_1^2} \quad \text{and} \quad Z_2 = \frac{(n_2 - 1)S_2^2}{\sigma_2^2},$$

then Z_1 and Z_2 are independent random variables with $Z_i \in \chi^2(n_i)$.

If

$$F = \frac{Z_1/(n_1 - 1)}{Z_2/(n_2 - 2)} = \frac{\frac{(n_1-1)S_1^2}{\sigma_1^2(n_1-1)}}{\frac{(n_2-1)S_2^2}{\sigma_2^2(n_2-1)}} = \frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2} = \frac{S_1^2 \sigma_2^2}{S_2^2 \sigma_1^2},$$

then F has the F-distribution with $(n_1 - 1)$ numerator degrees-of-freedom and $(n_2 - 1)$ denominator degrees-of-freedom. This is written as $F \in F(n_1 - 1, n_2 - 1)$. Problem #10 on page 28 asks you to verify this fact.