Stat 351 Fall 2007
Assignment #9 Solutions

1. (a) By Definition I, we see that $X_1 - \rho X_2$ is normally distributed with mean

$$E(X_1 - \rho X_2) = E(X_1) - \rho E(X_2) = 0$$

and variance

$$\operatorname{var}(X_1 - \rho X_2) = \operatorname{var}(X_1) + \rho^2 \operatorname{var}(X_2) - 2\rho \operatorname{cov}(X_1, X_2) = 1 + \rho^2 - 2\rho^2 = 1 - \rho^2.$$

That is, $X_1 - \rho X_2 = Y$ where $Y \in N(0, 1 - \rho^2)$. Hence, $Y = \sqrt{1-\rho^2} Z$ where $Z \in N(0,1)$. In other words, there exists a $Z \in N(0,1)$ such that

$$X_1 - \rho X_2 = \sqrt{1-\rho^2} Z.$$

1. (b) Since $\mathbf{X} = (X_1, X_2)'$ is MVN, and since

$$Z = \frac{X_1}{\sqrt{1-\rho^2}} - \frac{\rho X_2}{\sqrt{1-\rho^2}},$$

we conclude that $(Z, X_2)'$ is also a MVN. Hence, we know from Theorem V.7.1 that the components of a MVN are independent if and only if they are uncorrelated. We find

$$\begin{aligned} \operatorname{cov}(Z, X_2) = \operatorname{cov}\left(\frac{X_1}{\sqrt{1-\rho^2}} - \frac{\rho X_2}{\sqrt{1-\rho^2}}, X_2 \right) &= \frac{1}{\sqrt{1-\rho^2}} \operatorname{cov}(X_1, X_2) - \frac{\rho}{\sqrt{1-\rho^2}} \operatorname{var}(X_2) \\ &= \frac{\rho}{\sqrt{1-\rho^2}} - \frac{\rho}{\sqrt{1-\rho^2}} \\ &= 0 \end{aligned}$$

which verifies that Z and X_2 are, in fact, independent.

Exercise 4.2, page 127: If $\phi(t,u) = \exp\{it - 2t^2 - u^2 - tu\} = \exp\{it - \frac{1}{2}(4t^2 + 2tu + 2u^2)\}$, then we recognize this as the characteristic function of a normal random variable

$$\mathbf{X} = (X_1, X_2)' \in N\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 & 1 \\ 1 & 2 \end{pmatrix} \right).$$

Therefore, by Defintion I, $X_1 + X_2$ is normal with mean $E(X_1) + E(X_2) = 0 + 1 = 1$ and variance $\operatorname{var}(X_1 + X_2) = \operatorname{var}(X_1) + \operatorname{var}(X_2) + 2\operatorname{cov}(X_1, X_2) = 4 + 2 + 2 \cdot 1 = 8$. That is, $X_1 + X_2 \in N(1, 8)$.

Exercise 5.3, page 129: Let

$$B = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

so that $\mathbf{Y} = B\mathbf{X}$. By Theorem V.3.1, $\mathbf{Y}$ is MVN with mean

$$B\bar{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

and covariance matrix

$$B\mathbf{\Lambda}B' = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 7/2 & 1/2 & -1 \\ 1/2 & 1/2 & 0 \\ -1 & 0 & 1/2 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 3 \\ 0 & 3 & 5 \end{pmatrix}.$$

Hence, we see that $\mathbf{Y} \in \mathcal{N}(\overline{0}, \mathbf{\Sigma})$ where

$$\mathbf{\Sigma} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 3 \\ 0 & 3 & 5 \end{pmatrix}.$$

We now compute $\det[\mathbf{\Sigma}] = 10 - 9 = 1$ and

$$\mathbf{\Sigma}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & -3 \\ 0 & -3 & 2 \end{pmatrix}.$$

If we write $\mathbf{y} = (y_1, y_2, y_3)'$, then

$$\mathbf{y}'\mathbf{\Sigma}^{-1}\mathbf{y} = y_1^2 + 5y_2^2 - 6y_2y_3 + 2y_3^2$$

and so the density of $\mathbf{Y}$ is given by

$$f_{\mathbf{Y}}(\mathbf{y}) = \left(\frac{1}{2\pi}\right)^{3/2} \exp\left\{-\frac{1}{2}(y_1^2 + 5y_2^2 - 6y_2y_3 + 2y_3^2)\right\}.$$

Note that this problem could also be solved by observing that $Y_1 \in \mathcal{N}(0,1)$ and

$$(Y_2, Y_3)' \in \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}\right)$$

are independent so that $f_{\mathbf{Y}}(\mathbf{y}) = f_{Y_1}(y_1) \cdot f_{Y_2,Y_3}(y_2, y_3)$.

Problem #27, page 147: In order to determine the values of a and b for which $\mathbb{E}(U - a - bV)^2$ is a minimum, we must minimize the function $g(a,b) = \mathbb{E}(U - a - bV)^2$. If $U = X_1 + X_2 + X_3$ and $V = X_1 + 2X_2 + 3X_3$, then

$$U - a - bV = X_1 + X_2 + X_3 - a - b(X_1 + 2X_2 + 3X_3) = (1-b)X_1 + (1-2b)X_2 + (1-3b)X_3 - a.$$

Notice that $\mathbb{E}(U - a - bV)^2 = \operatorname{var}(U - a - bV) + [\mathbb{E}(U - a - bV)]^2$. We now compute

$$\begin{aligned} \operatorname{var}(U - a - bV) &= \operatorname{var}((1-b)X_1 + (1-2b)X_2 + (1-3b)X_3 - a) \\ &= (1-b)^2 \operatorname{var}(X_1) + (1-2b)^2 \operatorname{var}(X_2) + (1-3b)^2 \operatorname{var}(X_3) \\ &= (1-b)^2 + (1-2b)^2 + (1-3b)^2 \end{aligned}$$

using the fact that X_1, X_2, X_3 are i.i.d. $\mathcal{N}(1,1)$. Furthermore,

$$\begin{aligned} \mathbb{E}(U - a - bV) = \mathbb{E}((1-b)X_1 + (1-2b)X_2 + (1-3b)X_3 - a) &= (1-b) + (1-2b) + (1-3b) - a \\ &= 3 - 6b - a \end{aligned}$$

which implies that

$$g(a,b) = (1-b)^2 + (1-2b)^2 + (1-3b)^2 + [3-6b-a]^2 = 12 - 48b + 50b^2 - 6a + 12ab + a^2.$$

To minimize g, we begin by finding the critical points. That is,

$$\frac{\partial}{\partial a} g(a,b) = -6 + 12b + 2a = 0$$

implies $a + 6b = 3$, and

$$\frac{\partial}{\partial b} g(a,b) = -48 + 100b + 12a = 0$$

implies $25b + 3a = 12$. Solving the second equation for b yields

$$25b = 12 - 3a = 12 - 3(3 - 6b) \quad \text{and so} \quad b = \frac{3}{7}.$$

Substituting in gives

$$a = 3 - 6b = 3 - \frac{18}{7} = \frac{3}{7}.$$

Since

$$\frac{\partial^2}{\partial a^2} g(a,b) = 2 > 0$$

and

$$\frac{\partial^2}{\partial a^2} g(a,b) \cdot \frac{\partial^2}{\partial b^2} g(a,b) - \left(\frac{\partial^2}{\partial a \partial b} g(a,b) \right)^2 = 2 \cdot 100 - 12^2 = 56 > 0$$

we conclude by the second derivative test that $a = 3/7$, $b = 3/7$ is indeed the minimum.

3. (a) If $X \in U(0,1)$, then the distribution function of X is

$$F_X(x) = \begin{cases} 0, & \text{if } x \le 0, \\ x, & \text{if } 0 < x < 1, \\ 1, & \text{if } x \ge 1. \end{cases}$$

Therefore, if $Y = -\log X$, then the distribution function of Y is

$$F_Y(y) = P\{Y \le y\} = P\{-\log X \le y\} = P\{X \ge e^{-y}\} = 1 - P\{X \le e^{-y}\} = 1 - F_X(e^{-y}) = 1 - e^{-y}$$

provided that $y > 0$. We recognize this as the distribution function of an Exp(1) random variable. That is, $Y \in \text{Exp}(1)$.

3. (b) If $Y_i = -\log X_i$ for $i = 1, \ldots, n$, then by part **(a)**, we know that $Y_1, \ldots, Y_n$ are i.i.d. Exp(1) random variables. Furthermore,

$$\prod_{i=1}^n X_i = \exp\left\{ \log \prod_{i=1}^n X_i \right\} = \exp\left\{ \sum_{i=1}^n \log X_i \right\} = \exp\left\{ -\sum_{i=1}^n Y_i \right\}.$$

If we now let

$$Z = \sum_{i=1}^n Y_i,$$

then using characteristic functions it follows that $Z \in \Gamma(n, 1)$ (or it follows from Problem #20 in Chapter I or Problem #17 in Chapter IV). Finally, if we let

$$W = \prod_{i=1}^{n} X_i = e^{-Z},$$

then the distribution function of W is

$$F_W(w) = P\{W \leq w\} = P\{e^{-Z} \leq w\} = P\{Z \geq -\log w\} = 1 - P\{Z \leq -\log w\} = 1 - F_Z(-\log w).$$

Hence, the density function of W is

$$f_W(w) = \frac{1}{w} f_Z(-\log w) = \frac{1}{w} \cdot \frac{1}{\Gamma(n)} (-\log w)^{n-1} e^{\log w} = \frac{(-\log w)^{n-1}}{\Gamma(n)}, \quad 0 < w < 1.$$