Stat 351 Fall 2007
Assignment #6 Solutions

Problem #3, page 115: If $0 \le y \le 1/2$, then

$$f_Y(y) = \int_y^{1-y} f_{X_{(1)},X_{(2)}}(y,z)\,dz = \int_y^{1-y} 2\,dz = 2(1-2y).$$

On the other hand, if $1/2 \le y \le 1$, then

$$f_Y(y) = \int_{1-y}^{y} f_{X_{(1)},X_{(2)}}(z,1-y)\,dz = \int_{1-y}^{y} 2\,dz = 2(2y-1).$$

Problem #6, page 115: Since $E[F(X_{(n)}) - F(X_{(1)})] = E[F(X_{(n)})] - E[F(X_{(1)})]$, we compute each of $E[F(X_{(n)})]$ and $E[F(X_{(1)})]$ separately. Therefore, by definition,

$$E[F(X_{(n)})] = \int_{-\infty}^{\infty} F(y_n) f_{X_{(n)}}(y_n)\,dy_n.$$

From Theorem IV.1.2, we know that $f_{X_{(n)}}(y_n) = n[F_{X_{(n)}}(y_n)]^{n-1} f(y_n)$ so that

$$\int_{-\infty}^{\infty} F(y_n) f_{X_{(n)}}(y_n)\,dy_n = \int_{-\infty}^{\infty} n[F(y_n)]^n f(y_n)\,dy_n.$$

Making the substitution $u = F(y_n)$ so that $du = F'(y_n)dy_n = f(y_n)dy_n$ gives

$$\int_{-\infty}^{\infty} n[F(y_n)]^n f(y_n)\,dy_n = \int_0^1 nu^n\,du = \frac{n}{n+1}.$$

Note that since F is a distribution, our new limits of integration are $F(-\infty) = 0$ and $F(\infty) = 1$. As for $E[F(X_{(1)})]$, using Theorem IV.1.2, we compute

$$E[F(X_{(1)})] = \int_{-\infty}^{\infty} F(y_1) f_{X_{(1)}}(y_1)\,dy_1 = \int_{-\infty}^{\infty} F(y_1) n[1-F(y_1)]^{n-1} f(y_1)\,dy_1.$$

Making the same substitution as above gives

$$\int_{-\infty}^{\infty} F(y_1) n[1-F(y_1)]^{n-1} f(y_1)\,dy_1 = \int_0^1 nu(1-u)^{n-1}\,du = n\int_0^1 (1-v)v^{n-1}\,dv = 1 - \frac{n}{n+1}.$$

Finally, we combine our two results to conclude that

$$E[F(X_{(n)}) - F(X_{(1)})] = \frac{n}{n+1} - \left[1 - \frac{n}{n+1}\right] = \frac{n-1}{n+1}.$$

Problem #9, page 116: **(a)**: If X_1 and X_2 are independent Exp(a) random variables, then by Theorem IV.2.1, the joint density of $(X_{(1)}, X_{(2)})$ is given by

$$f_{X_{(1)},X_{(2)}}(y_1,y_2) = \begin{cases} \frac{2}{a^2}\exp\left(-\frac{y_1+y_2}{a}\right), & \text{for } 0 < y_1 < y_2 < \infty, \\ 0, & \text{otherwise.} \end{cases}$$

Suppose that $U = X_{(1)}$ and let $V = X_{(2)} - X_{(1)}$. Solving for $X_{(1)}$ and $X_{(2)}$ gives

$$X_{(1)} = U \quad \text{and} \quad X_{(2)} = U + V.$$

The Jacobian of this transformation is given by

$$J = \begin{vmatrix} \frac{\partial y_1}{\partial u} & \frac{\partial y_1}{\partial v} \\ \frac{\partial y_2}{\partial u} & \frac{\partial y_2}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1.$$

Therefore, by Theorem I.2.1, the density of (U, V) is given by

$$f_{U,V}(u, v) = f_{X_{(1)},X_{(2)}}(u, u+v) \cdot |J| = \frac{2}{a^2} \exp\left(-\frac{u+u+v}{a}\right) = \frac{2}{a^2} \exp\left(-\frac{2u+v}{a}\right) = \frac{2}{a} e^{-2u/a} \cdot \frac{1}{a} e^{-v/a}$$

provided that $v > 0$ and $u > 0$. The marginal density of U is

$$f_U(u) = \int_{-\infty}^{\infty} f_{U,V}(u, v)\, dv = \int_0^{\infty} \frac{2}{a} e^{-2u/a} \cdot \frac{1}{a} e^{-v/a}\, dv = \frac{2}{a} e^{-2u/a}$$

for $u > 0$. We recognize that this is the density of an exponential random variable with parameter $a/2$; that is, $U = X_{(1)} \in \text{Exp}(a/2)$. The marginal density of V is

$$f_V(v) = \int_{-\infty}^{\infty} f_{U,V}(u, v)\, du = \int_0^{\infty} \frac{2}{a} e^{-2u/a} \cdot \frac{1}{a} e^{-v/a}\, du = \frac{1}{a} e^{-v/a}$$

for $v > 0$. We recognize that this is the density of an exponential random variable with parameter a; that is, $V = X_{(2)} - X_{(1)} \in \text{Exp}(a)$. Since we can express $f_{U,V}(u, v) = f_U(u) \cdot f_V(v)$ we conclude that U and V are independent; in other words, $X_{(1)}$ and $X_{(2)} - X_{(1)}$ are independent.

(b): To compute $E(X_{(2)}|X_{(1)} = y)$, we can use properties of conditional expectation (Theorem II.2.2):

$$\begin{aligned} E(X_{(2)}|X_{(1)} = y) &= E(X_{(2)} - X_{(1)} + X_{(1)}|X_{(1)} = y) \\ &= E(X_{(2)} - X_{(1)}|X_{(1)} = y) + E(X_{(1)}|X_{(1)} = y) \\ &= E(X_{(2)} - X_{(1)}) + y \\ &= a + y \end{aligned}$$

where the first expression after the third equality follows since $X_{(2)} - X_{(1)}$ is independent of $X_{(1)}$ and the second expression follows since $X_{(1)}$ is "known" when conditioned on the value $X_{(1)} = y$.

As for $E(X_{(1)}|X_{(2)} = x)$, we need to compute this by definition of conditional expectation. That is,

$$f_{X_{(1)}|X_{(2)}=x}(y_1) = \frac{f_{X_{(1)},X_{(2)}}(y_1, x)}{f_{X_{(2)}}(x)} = \frac{\frac{2}{a^2} e^{-y_1/a} \cdot e^{-x/a}}{\frac{2}{a}(1 - e^{-x/a}) \cdot e^{-x/a}} = \frac{1}{a} \frac{e^{-y_1/a}}{1 - e^{-x/a}}$$

provided $0 < y_1 < x$. This then gives

$$E(X_{(1)}|X_{(2)} = x) = \int_{-\infty}^{\infty} f_{X_{(1)}|X_{(2)}=x}(y_1)\, dy_1 = \int_0^x \frac{y_1}{a} \frac{e^{-y_1/a}}{1 - e^{-x/a}}\, dy_1 = \frac{1}{a(1 - e^{-x/a})} \int_0^x y_1\, e^{-y_1/a}\, dy_1.$$

Integrating by parts gives

$$\int_0^x y_1\, e^{-y_1/a}\, dy_1 = a^2 - a^2 e^{-x/a} - axe^{-x/a}.$$

Therefore,

$$E(X_{(1)}|X_{(2)} = x) = \frac{a^2 - a^2e^{-x/a} - axe^{-x/a}}{a(1-e^{-x/a})} = a - \frac{xe^{-x/a}}{1-e^{-x/a}} = a - \frac{x}{e^{x/a}-1}.$$

Problem #10, page 116: Let X_1, X_2, and X_3 are independent, identically distributed $U(0,1)$ random variables. Notice that if $x > 1/2$, then since $X_{(3)} > X_{(1)}$ we conclude

$$P(X_{(3)} > \tfrac{1}{2} \,|\, X_{(1)} = x) = 1.$$

On the other hand, suppose that $0 \le x \le 1/2$. By equation (3.10) on page 114,

$$f_{X_{(1)},X_{(3)}}(y_1, y_3) = 6(y_3 - y_1)$$

provided $0 < y_1 < y_3 < 1$. Therefore, we find

$$P(X_{(3)} > \tfrac{1}{2} \,|\, X_{(1)} = x) = \frac{\displaystyle\int_{1/2}^{1} f_{X_{(1)},X_{(3)}}(x, y_3)\, dy_3}{f_{X_{(1)}}(x)}.$$

For the numerator we calculate

$$\int_{1/2}^{1} f_{X_{(1)},X_{(3)}}(x, y_3)\, dy_3 \int_{1/2}^{1} 6(y_3 - x)\, dy_3 = (3y_3^2 - 6xy_3)\Big|_{1/2}^{1} = \frac{9}{4} - 3x = \frac{3}{4}(3-4x).$$

As for the denominator, from Remark 3.1 on page 114, we find

$$f_{X_{(1)}}(x) = 3(1-x)^2$$

provided $0 < x < 1$. Putting these pieces together, we conclude

$$P(X_{(3)} > \tfrac{1}{2} \,|\, X_{(1)} = x) = \frac{\frac{3}{4}(3-4x)}{3(1-x)^2} = \frac{(3-4x)}{4(1-x)^2}.$$

That is,

$$P(X_{(3)} > \tfrac{1}{2} \,|\, X_{(1)} = x) = \begin{cases} \frac{(3-4x)}{4(1-x)^2}, & \text{if } 0 \le x \le 1/2, \\ 1, & \text{if } x > 1/2. \end{cases}$$

Problem #12, page 116: Since $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ are i.i.d. $U(0,a)$ random variables, we conclude from Theorem IV.1.2 that $X_{(n)}$ and $Y_{(n)}$ are independent and identically distributed $\beta(1,n)$ random variables. In order to simplify matters we let $X = X_{(n)}$ and $Y = Y_{(n)}$ so that X and Y have common density function

$$f(x) = \frac{n}{a^n}x^{n-1}, \quad 0 < x < a$$

and common distribution function

$$F(x) = \begin{cases} 0, & x \le 0, \\ \frac{x^n}{a^n}, & 0 < x < a, \\ 1, & x \ge 1. \end{cases}$$

If we now let $S = \min\{X, Y\}$ and $T = \max\{X, Y\}$, then Theorem IV.2.1 implies that the joint density of (S, T) is

$$f_{S,T}(s,t) = 2 \cdot \frac{n}{a^n} s^{n-1} \cdot \frac{n}{a^n} t^{n-1} = \frac{2n^2}{a^{2n}} s^{n-1} t^{n-1}, \quad 0 < s < t < a.$$

The next step is to let $U = \frac{T}{S}$ and $V = S$ so that $S = V$ and $T = UV$. We find the Jacobian of this transformation is

$$J = \begin{vmatrix} \frac{\partial s}{\partial u} & \frac{\partial s}{\partial v} \\ \frac{\partial t}{\partial u} & \frac{\partial t}{\partial v} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ v & u \end{vmatrix} = -v.$$

The density of (U, V) is therefore given by

$$f_{U,V}(u,v) = f_{S,T}(v, uv) \cdot |J| = \frac{2n^2}{a^{2n}} v^{n-1} (uv)^{n-1} \cdot v = \frac{2n^2}{a^{2n}} u^{n-1} v^{2n-1}$$

provided that $1 < u < \infty$, $0 < v < \frac{a}{u} < a$. The marginal density for U is therefore given by

$$f_U(u) = \int_{-\infty}^{\infty} f_{U,V}(u,v)\,\mathrm{d}v = \frac{2n^2}{a^{2n}} u^{n-1} \int_0^{a/u} v^{2n-1}\,\mathrm{d}v = \frac{n}{a^{2n}} u^{n-1} v^{2n} \Big|_{v=0}^{v=a/u} = \frac{n}{a^{2n}} u^{n-1} \frac{a^{2n}}{u^{2n}} = n u^{-(n+1)}$$

provided that $1 < u < \infty$. Since we are interested in

$$Z_n = n \log \left(\frac{\max\{X_{(n)}, Y_{(n)}\}}{\min\{X_{(n)}, Y_{(n)}\}} \right) = n \log U$$

we can now use techniques from Chapter I to find the density of Z_n. Let $Z = Z_n = n \log U$. Therefore, $F_Z(z) = P(Z \le z) = P(U \le e^{z/n})$ and so

$$f_Z(z) = \frac{1}{n} e^{z/n} f_U(e^{z/n}) = \frac{1}{n} e^{z/n} \cdot n (e^{z/n})^{-(n+1)} = e^{-z}$$

provided that $0 < z < \infty$. Hence we conclude that $Z_n \in \mathrm{Exp}(1)$.