Statistics 351–Probability I
Fall 2006 (200630)
Final Exam Solutions

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1. (a) Solving for X and Y gives $X = UV$ and $Y = V - UV$, so that the Jacobian of this transformation is

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ -v & 1-u \end{vmatrix} = v - vu + vu = v.$$

By Theorem I.2.1, the joint density of $(U, V)'$ is therefore given by

$$\begin{aligned} f_{U,V}(u,v) = f_{X,Y}(uv, uv - uv) \cdot |J| &= \frac{\theta^{-\alpha-\beta}}{\Gamma(\alpha)\Gamma(\beta)} (uv)^{\alpha-1} (v - uv)^{\beta-1} \exp\left\{-\frac{v}{\theta}\right\} v \\ &= \frac{\theta^{-\alpha-\beta}}{\Gamma(\alpha)\Gamma(\beta)} u^{\alpha-1}(1-u)^{\beta-1} v^{\alpha+\beta-1} \exp\left\{-\frac{v}{\theta}\right\} \\ &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} u^{\alpha-1}(1-u)^{\beta-1} \cdot \frac{\theta^{-\alpha-\beta}}{\Gamma(\alpha+\beta)} v^{\alpha+\beta-1} \exp\left\{-\frac{v}{\theta}\right\} \end{aligned}$$

provided that $0 < u < 1$ and $0 < v < \infty$.

1. (b) We recognize that the joint density for U and V can be factored as a product of the densities for U and V, respectively. Thus,

$$f_U(u) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} u^{\alpha-1}(1-u)^{\beta-1}, \quad 0 < u < 1$$

which we recognize as the density of a Beta(α, β) random variable.

2. (a) We see that $f_{X,Y}(x,y) \ge 0$ for all x, y, and that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx\, dy = \int_0^1 \int_0^x 12y^2\, dy\, dx = \int_0^1 4x^3\, dx = x^4 \Big|_0^1 = 1.$$

Thus, $f_{X,Y}$ is a legitimate density.

2. (b) We compute

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy = \int_0^x 12y^2\, dy = 4x^3, \quad 0 < x < 1.$$

2. (c) We compute

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx = \int_y^1 12y^2\, dx = 12y^2(1-y), \quad 0 < y < 1.$$

2. (d) We compute

$$f_{X|Y=y}(x) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{12y^2}{12y^2(1-y)} = \frac{1}{1-y}, \quad y < x < 1.$$

2. (e) We compute

$$f_{Y|X=x}(y) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{12y^2}{4x^3} = \frac{3y^2}{x^3}, \quad 0 < y < x.$$

2. (f) We compute

$$E(X) = \int_{-\infty}^{\infty} x f_X(x)\, dx = \int_0^1 x \cdot 4x^3\, dx = \frac{4}{5}.$$

2. (g) We compute

$$E(Y|X=x) = \int_{-\infty}^{\infty} y f_{Y|X=x}(y)\, dy = \int_0^x y \cdot \frac{3y^2}{x^3}\, dy = \frac{3x^4}{4x^3} = \frac{3}{4}x.$$

2. (h) Using properties of conditional expectation (Theorem II.2.1), we compute

$$E(Y) = E(\,E(Y|X)\,) = E\left(\frac{3}{4}X\right) = \frac{3}{4}E(X) = \frac{3}{4}\cdot\frac{4}{5} = \frac{3}{5}.$$

2. (i) Solution 1: Using properties of conditional expectation (Theorem II.2.2) gives

$$E(XY) = E(\,E(XY|X)\,) = E(X\,E(Y|X)\,) = E\left(X \cdot \frac{3}{4}X\right) = \frac{3}{4}E(X^2).$$

Since

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx = \int_0^1 x^2 \cdot 4x^3\, dx = \frac{4}{6} = \frac{2}{3},$$

we conclude

$$E(XY) = \frac{3}{4}\cdot\frac{2}{3} = \frac{1}{2}.$$

Solution 2: By definition,

$$\begin{aligned} E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{X,Y}(x,y)\, dx\, dy &= \int_0^1\int_0^x xy \cdot 12y^2\, dy\, dx = 12\int_0^1 x \int_0^x y^3\, dy\, dx \\ &= 12\int_0^1 x \cdot \frac{1}{4}x^4\, dx = 3\int_0^1 x^5\, dx = \frac{3}{6} = \frac{1}{2}. \end{aligned}$$

3. (a) Let

$$B = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

so that $\mathbf{Y} = B\mathbf{X}$. By Theorem V.3.1, $\mathbf{Y}$ is MVN with mean

$$B\boldsymbol{\mu} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and covariance matrix

$$B\boldsymbol{\Lambda}B' = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & -1 \\ -1 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 12 & -3 \\ -3 & 1 \end{pmatrix}.$$

3. (b) Note that

$$\det\begin{pmatrix}12 & -3\\ -3 & 1\end{pmatrix} = 12 - 9 = 3$$

so that

$$\begin{pmatrix}12 & -3\\ -3 & 1\end{pmatrix}^{-1} = \begin{pmatrix}\frac{1}{3} & 1\\ 1 & 4\end{pmatrix}.$$

Thus, we can conclude

$$f_{Y_1,Y_2}(y_1,y_2) = \frac{1}{2\pi}\cdot\frac{1}{\sqrt{3}}\exp\left\{-\frac{1}{2}\left(\frac{1}{3}y_1^2 + 2y_1y_2 + 4y_2^2\right)\right\}.$$

4. (a) We recognize $f_{\mathbf{X}}(x,y)$ as the density function of a multivariate normal random variable with mean

$$\boldsymbol{\mu} = \begin{pmatrix}0\\0\end{pmatrix}$$

and covariance matrix $\mathbf{\Lambda}$ where

$$\mathbf{\Lambda}^{-1} = \begin{pmatrix}1/2 & -1/2\\ -1/2 & 1\end{pmatrix}.$$

Inverting this matrix gives

$$\mathbf{\Lambda} = \begin{pmatrix}4 & 2\\ 2 & 2\end{pmatrix}.$$

That is,

$$\mathbf{X} \in N\left(\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}4 & 2\\ 2 & 2\end{pmatrix}\right).$$

4. (b) The characteristic function of $\mathbf{X}$ is

$$\varphi_{\mathbf{X}}(t_1,t_2) = \exp\left\{-\frac{1}{2}(4t_1^2 + 4t_1t_2 + 2t_2^2)\right\}.$$

4. (c) Recall that since $\mathbf{X} = (X,Y)'$ is multivariate normal, the distribution of $Y|X = x$ is normal with mean $\mu_y + \rho\frac{\sigma_y}{\sigma_x}(x - \mu_x)$ and variance $\sigma_y^2(1-\rho^2)$ where $\rho = \operatorname{corr}(X,Y)$. From **(a)**, we know that $\mu_y = \mu_x = 0$, $\sigma_y = \sqrt{2}$, $\sigma_x = 2$, and $\rho = \frac{\operatorname{cov}(X,Y)}{\sigma_x\sigma_y} = \frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}}$. Therefore,

$$\mu_y + \rho\frac{\sigma_y}{\sigma_x}(x-\mu_x) = 0 + \frac{1}{\sqrt{2}}\cdot\frac{\sqrt{2}}{2}(x-0) = \frac{x}{2} \quad\text{and}\quad \sigma_y^2(1-\rho^2) = 2\left(1 - \left(\frac{1}{\sqrt{2}}\right)^2\right) = 1$$

so that $Y|X = x \in N(\frac{x}{2}, 1)$.

5. By definition, $f_{X,Y}(x,y) = f_{Y|X=x}(y)f_X(x)$ so that

$$f_{X,Y}(x,y) = \frac{1}{\sqrt{2\pi}}e^{-\frac{(y-x)^2}{2}}\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}} = \frac{1}{2\pi}\exp\left\{-\frac{1}{2}\left((y-x)^2 + x^2\right)\right\} = \frac{1}{2\pi}\exp\left\{-\frac{1}{2}\left(2x^2 - 2xy + y^2\right)\right\}$$

which we recognize as the density function of a multivariate normal random variable with mean

$$\boldsymbol{\mu} = \begin{pmatrix}0\\0\end{pmatrix}$$

(continued)

and covariance matrix $\mathbf{\Lambda}$ where

$$\mathbf{\Lambda}^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}.$$

Inverting this matrix gives

$$\mathbf{\Lambda} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

That is,

$$(X, Y)' \in N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}\right).$$

Thus, we conclude that $Y \in N(0, 2)$.

6. (a) In order to find the eigenvalues of $\mathbf{\Lambda}$, we must find those values of λ such that $\det(\mathbf{\Lambda} - \lambda I) = 0$. Therefore,

$$\det(\mathbf{\Lambda} - \lambda I) = \begin{pmatrix} 6-\lambda & -5 \\ -5 & 6-\lambda \end{pmatrix} = (6-\lambda)^2 - 25 = \lambda^2 - 12\lambda + 36 - 25 = \lambda^2 - 12\lambda + 11 = (\lambda - 11)(\lambda - 1)$$

so that the eigenvalues of $\mathbf{\Lambda}$ are $\lambda_1 = 11$ and $\lambda_2 = 1$.

6. (b) Since $\lambda_1 = 11$,

$$(\mathbf{\Lambda} - \lambda_1 I \,|\, 0) = \left(\begin{array}{cc|c} -5 & -5 & 0 \\ -5 & -5 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

and since $\lambda_2 = 1$,

$$(\mathbf{\Lambda} - \lambda_2 I \,|\, 0) = \left(\begin{array}{cc|c} 5 & -5 & 0 \\ -5 & 5 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

we conclude that eigenvectors for λ_1 and λ_2 are

$$\mathbf{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

respectively. Therefore, the diagonal matrix is

$$D = \operatorname{diag}(\lambda_1, \lambda_2) = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 11 & 0 \\ 0 & 1 \end{pmatrix}$$

and the orthogonal matrix is

$$C = \left(\frac{\mathbf{v}_1}{||\mathbf{v}_1||} \; \frac{\mathbf{v}_2}{||\mathbf{v}_2||} \right) = \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

since $||\mathbf{v}_1|| = ||\mathbf{v}_2|| = \sqrt{2}$.

6. (c) If $\mathbf{Y} = C'\mathbf{X}$, then by Theorem V.3.1, $\mathbf{Y}$ is MVN with mean $C'\boldsymbol{\mu}$ and covariance matrix $C'\mathbf{\Lambda}C'' = C'\mathbf{\Lambda}C = D$ using our result from **(b)**. Hence, we conclude

$$\mathbf{Y} \in N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 11 & 0 \\ 0 & 1 \end{pmatrix}\right).$$

6. (d) Since $\mathbf{Y}$ is multivariate normal we know from Definition I that Y_1 and Y_2 are each one-dimensional normals. We also know from Theorem V.7.1 that the components of $\mathbf{Y}$ are independent if and only if they are uncorrelated. From **(c)** we know that $\operatorname{cov}(Y_1, Y_2) = 0$ so that Y_1 and Y_2 are, in fact, independent.

7. Observe that since $f_{X,Y}(x,y) = 2x$ for $0 < x < 1$, $0 < y < 1$, we can immediately conclude that X and Y are independent with $f_X(x) = 2x$, $0 < x < 1$, and $f_Y(y) = 1$, $0 < y < 1$. Therefore, using the law of total probability,

$$P(X^2 < Y < X) = \int_0^1 P(x^2 < Y < x | X = x) f_X(x) \, dx = \int_0^1 P(x^2 < Y < x) f_X(x) \, dx$$

since $P(x^2 < Y < x | X = x) = P(x^2 < Y < x)$ by the independence of X and Y. Now,

$$P(x^2 < Y < x) = \int_{x^2}^{x} f_Y(y) \, dy = \int_{x^2}^{x} 1 \, dy = x - x^2$$

so that

$$\int_0^1 P(x^2 < Y < x) f_X(x) \, dx = \int_0^1 (x - x^2) f_X(x) \, dx = \int_0^1 2x(x - x^2) \, dx = \frac{2}{3} - \frac{2}{4} = \frac{1}{6}.$$

8. (a) Let $U = \frac{X_{(2)} - X_{(1)}}{2}$ and $V = \frac{X_{(2)} + X_{(1)}}{2}$ so that $U = X_M$ and $V = \overline{X}$. Solving for $X_{(1)}$ and $X_{(2)}$ we find

$$X_{(1)} = V - U \quad \text{and} \quad X_{(2)} = V + U.$$

The Jacobian of this transformation is

$$J = \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = -2.$$

Since

$$f_{X_{(1)},X_{(2)}}(y_1, y_2) = 2! \, e^{-y_1} e^{-y_2}, \quad 0 < y_1 < y_2 < \infty,$$

we find from Theorem I.2.1 that the joint density of (U, V) is therefore given by

$$f_{U,V}(u,v) = f_{X_{(1)},X_{(2)}}(v - u, v + u) \cdot |J| = 2! \, e^{-(v-u)} e^{-(v+u)} \cdot 2 = 4e^{-2v}$$

provided that $0 < u < v < \infty$. In other words, the joint density of the sample median X_M and the sample mean $\overline{X}$ is

$$f_{X_M,\overline{X}}(u,v) = 4e^{-2v}, \quad 0 < u < v < \infty.$$

8. (b) The density of the sample median X_M is given by

$$f_{X_M}(u) = \int_u^\infty f_{X_M,\overline{X}}(u,v) \, dv = \int_u^\infty 4e^{-2v} \, dv = -2e^{-2v} \Big|_u^\infty = 2e^{-2u}$$

provided that $0 < u < \infty$. That is, $X_M \in \text{Exp}(1/2)$.

8. (c) The density of the sample mean $\overline{X}$ is given by

$$f_{\overline{X}}(v) = \int_0^v f_{X_M,\overline{X}}(u,v) \, du = \int_0^v 4e^{-2v} \, du = 4ve^{-2v}$$

provided that $0 < v < \infty$. That is, $\overline{X} \in \Gamma(2, 1/2)$.

9. (a) Recall that since $\mathbf{X} = (X, Y)'$ is multivariate normal, the distribution of $Y | X = x$ is normal with mean $\mu_y + \rho \frac{\sigma_y}{\sigma_x}(x - \mu_x)$ and variance $\sigma_y^2(1 - \rho^2)$ where $\rho = \text{corr}(X, Y)$. Thus, since $\mu_x = \mu_y = 0$ and $\sigma_x = \sigma_y = 1$, we find that $\rho = \text{corr}(X, Y) = \text{cov}(X, Y)$ and we conclude $E(Y|X) = \rho X$ so that

$$\text{cov}(X, Y - E(Y|X)) = \text{cov}(X, Y - \rho X) = \text{cov}(X, Y) - \rho \, \text{cov}(X, X) = \rho - \rho \cdot \text{var}(X) = \rho - \rho \cdot 1 = 0.$$

Hence, X and $Y - E(Y|X)$ are uncorrelated.

9. (b) Since $\mathbf{X} = (X, Y)'$ is multivariate normal, we know from Definition I that any linear combination of the components of $\mathbf{X}$ must be a one-dimensional normal. In particular, this means that $Y - \rho X = Y - E(Y|X)$ is normal. Since X is also normal, and since we know from Theorem V.7.1 that the components of a multivariate normal are uncorrelated if and only if they are independent, we conclude that X and $Y - E(Y|X)$ must be independent (since we showed in **(a)** that they are uncorrelated).

10. (a) Since $X_4 \in \text{Po}(4)$, we find

$$P(X_4 = j) = \frac{4^j}{j!}e^{-4}, \quad j = 1, 2.$$

10. (b) Using the definition of conditional probability and the fact that increments of the Poisson process are independent, we have

$$\begin{aligned} P(X_4 = j|X_3 = 1) = \frac{P(X_4 = j, X_3 = 1)}{P(X_3 = 1)} = \frac{P(X_4 - X_3 = j - 1, X_3 = 1)}{P(X_3 = 1)} &= \frac{P(X_4 - X_3 = j - 1)P(X_3 = 1)}{P(X_3 = 1)} \\ &= P(X_4 - X_3 = j - 1). \end{aligned}$$

Since $X_4 - X_3 \in \text{Po}(1)$, we find

$$P(X_4 = j|X_3 = 1) = P(X_4 - X_3 = j - 1) = \frac{1^{j-1}}{(j-1)!}e^{-1} = \frac{e^{-1}}{(j-1)!}, \quad j = 1, 2.$$

10. (c) Using the definition of conditional probability and the fact that increments of the Poisson process are independent, we have

$$P(X_1 = 0|X_3 = 1) = \frac{P(X_1 = 0, X_3 = 1)}{P(X_3 = 1)} = \frac{P(X_3 - X_1 = 1, X_1 = 0)}{P(X_3 = 1)} = \frac{P(X_3 - X_1 = 1)P(X_1 = 0)}{P(X_3 = 1)}.$$

Since $X_3 - X_1 \in \text{Po}(2)$, $X_3 \in \text{Po}(3)$, and $X_1 \in \text{Po}(1)$, we find

$$P(X_1 = 0|X_3 = 1) = \frac{\frac{2^1}{1!}e^{-2} \cdot \frac{1^0}{0!}e^{-1}}{\frac{3^1}{1!}e^{-3}} = \frac{2}{3}.$$

10. (d) By adding and subtracting X_3, we compute

$$\text{cov}(X_3, X_4) = \text{cov}(X_3, X_4 - X_3 + X_3) = \text{cov}(X_3, X_4 - X_3) + \text{cov}(X_3, X_3) = 0 + \text{var}(X_3)$$

using the fact that the increments $X_4 - X_3$ and X_3 are independent. Since $X_3 \in \text{Po}(3)$ we know $\text{var}(X_3) = 3$ so that

$$\text{cov}(X_3, X_4) = \text{var}(X_3) = 3.$$

10. (e) By adding and subtracting X_1, we compute

$$E(X_3|X_1 = j) = E(X_3 - X_1 + X_1|X_1 = j) = E(X_3 - X_1|X_1 = j) + E(X_1|X_1 = j) = E(X_3 - X_1) + j$$

where we have used the facts that $E(X_3 - X_1|X_1 = j) = E(X_3 - X_1)$ since $X_3 - X_1$ and X_1 are independent, and $E(X_1|X_1 = j) = j$ by "taking out what is known." (See Theorems II.2.1 and II.2.2.) Since $X_3 - X_1 \in \text{Po}(2)$ we know $E(X_3 - X_1) = 2$ so that

$$E(X_3|X_1 = j) = 2 + j, \quad j = 0, 1, 2, \ldots.$$

11. (a) Let T_8 denote the time after waking at which Keith lights his 8th cigarette. Since $T_8 \in \Gamma(8, \frac{1}{4})$, we conclude

$$E(T_8) = 8 \cdot \frac{1}{4} = 2$$

so that he is expected to light his 8th cigarette at noon, namely 2 hours after 10:00 a.m.

11. (b) The probability that he lights 3 cigarettes or more between noon and 1:00 p.m. is

$$\begin{aligned} P(X_3 - X_2 \geq 3) &= 1 - P(X_3 - X_2 < 3) \\ &= 1 - P(X_3 - X_2 = 0) - P(X_3 - X_2 = 1) - P(X_3 - X_2 = 2) \\ &= 1 - \frac{4^0}{0!}e^{-4} - \frac{4^1}{1!}e^{-4} - \frac{4^2}{2!}e^{-4} \\ &= 1 - 13e^{-4} \end{aligned}$$

since $X_3 - X_2 \in \text{Po}(4)$.