Stat 351 Fall 2006
Assignment #8 Solutions

**Problem #9, page 144**: Note that by Theorem 7.1, in order to show $X_1$, $X_2$, and $X_3$ are independent, it is enough to show that $\operatorname{cov}(X_1, X_2) = \operatorname{cov}(X_1, X_3) = \operatorname{cov}(X_2, X_3) = 0$. Thus, if $X_1$ and $X_2 + X_3$ are independent, then $\operatorname{cov}(X_1, X_2 + X_3) = \operatorname{cov}(X_1, X_2) + \operatorname{cov}(X_1, X_3) = 0$ and so

$$\operatorname{cov}(X_1, X_2) = -\operatorname{cov}(X_1, X_3). \tag{1}$$

If $X_2$ and $X_1 + X_3$ are independent, then $\operatorname{cov}(X_2, X_1 + X_3) = \operatorname{cov}(X_2, X_1) + \operatorname{cov}(X_2, X_3) = 0$ and so

$$\operatorname{cov}(X_2, X_1) = -\operatorname{cov}(X_2, X_3). \tag{2}$$

Finally, if $X_3$ and $X_1+X_2$ are independent, then $\operatorname{cov}(X_3, X_1+X_2) = \operatorname{cov}(X_3, X_1)+\operatorname{cov}(X_3, X_2) = 0$ and so

$$\operatorname{cov}(X_3, X_1) = -\operatorname{cov}(X_3, X_2). \tag{3}$$

Since (1), (2), and (3) must be simultaneously satisfied, the only possibility is that $\operatorname{cov}(X_1, X_2) = \operatorname{cov}(X_1, X_3) = \operatorname{cov}(X_2, X_3) = 0$. Hence, $X_1$, $X_2$, and $X_3$ are independent as required.

**Problem #10, page 145**: From Assignment #7, we know that the distribution of $\mathbf{Y} = (Y_1, Y_2)'$ is

$$\mathbf{Y} \in N\left(\begin{pmatrix} 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 10 & 5 \\ 5 & 5 \end{pmatrix}\right)$$

and so we see that $Y_1 \in N(2, 10)$, $Y_2 \in N(-1, 5)$, and $\operatorname{corr}(Y_1, Y_2) = \frac{1}{\sqrt{2}}$. Thus, by the results in Section V.6, the distribution of $Y_1|Y_2 = y$ is normal with mean $2 + \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{10}}{\sqrt{5}}(y - (-1)) = y + 3$ and variance $10\left(1 - \left(\frac{1}{\sqrt{2}}\right)^2\right) = 5$. That is,

$$Y_1|Y_2 = y \in N(y + 3, 5).$$

**Problem #11, page 145**: From Assignment #7, we know that the distribution of $\mathbf{Y} = (Y_1, Y_2)'$ is

$$\mathbf{Y} \in N\left(\begin{pmatrix} 0 \\ 8 \end{pmatrix}, \begin{pmatrix} 16 & -2 \\ -2 & 16 \end{pmatrix}\right)$$

and so we see that $Y_1 \in N(0, 16)$, $Y_2 \in N(8, 16)$, and $\operatorname{corr}(Y_1, Y_2) = -\frac{1}{8}$. Thus, by the results in Section V.6, the distribution of $Y_1|Y_2 = 10$ is normal with mean $0 - \frac{1}{8} \cdot \frac{4}{4}(10 - 8) = -\frac{1}{4}$ and variance $16\left(1 - \left(-\frac{1}{8}\right)^2\right) = \frac{63}{4}$. That is,

$$Y_1|Y_2 = 10 \in N\left(-\tfrac{1}{4}, \tfrac{63}{4}\right).$$

**Problem #13, page 145**: From Assignment #7, we know that the distribution of $\mathbf{X} = (X_1, X_2, X_3)'$ is

$$\mathbf{X} \in N\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 & 4 & -5 \\ 4 & 9 & -10 \\ -5 & -10 & 13 \end{pmatrix}\right)$$

and so we see that $X_1 \in N(0, 2)$, $X_2 \in N(0, 9)$, and $X_3 \in N(0, 13)$. Since $\operatorname{cov}(X_1, X_3) = -5$, we conclude that $X_1 + X_3 \in N(0, 5)$. Finally, we compute $\operatorname{cov}(X_2, X_1 + X_3) = \operatorname{cov}(X_2, X_1) + \operatorname{cov}(X_2, X_3) = 4 - 10 = -6$ and so $\operatorname{corr}(X_2, X_1 + X_3) = -\frac{2}{\sqrt{5}}$. Thus, by the results in Section V.6, the distribution of $X_2|X_1+X_3=x$ is normal with mean $0-\frac{2}{\sqrt{5}}\cdot\frac{3}{\sqrt{5}}(x-0)=-\frac{6x}{5}$ and variance $9\left(1-\left(-\frac{2}{\sqrt{5}}\right)^2\right)=\frac{9}{5}$. That is,

$$X_2|X_1+X_3=x \in N\left(-\tfrac{6x}{5},\tfrac{9}{5}\right).$$

**Problem #14, page 145**: From Assignment #7, we know that the distribution of $\mathbf{Y}=(Y_1,Y_2,Y_3)'$ is

$$\mathbf{Y}\in N\left(\begin{pmatrix}0\\0\\0\end{pmatrix},\begin{pmatrix}2&1&1\\1&2&1\\1&1&2\end{pmatrix}\right).$$

By definition,

$$f_{Y_1|Y_2=0,Y_3=0}(y)=\frac{f_{Y_1,Y_2,Y_3}(y,0,0)}{f_{Y_2,Y_3}(0,0)}.$$

From Definition III, we know

$$f_{Y_1,Y_2,Y_3}(y,0,0)=\left(\frac{1}{2\pi}\right)^{3/2}\frac{1}{\sqrt{4}}\,e^{-\frac{1}{2}\frac{3}{4}y^2}$$

since

$$\begin{pmatrix}2&1&1\\1&2&1\\1&1&2\end{pmatrix}^{-1}=\frac{1}{4}\begin{pmatrix}3&-1&-1\\-1&3&-1\\-1&-1&3\end{pmatrix}.$$

The joint distribution of $(Y_2,Y_3)'$ is

$$(Y_2,Y_3)'\in N\left(\begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}2&1\\1&2\end{pmatrix}\right)$$

and so

$$f_{Y_2,Y_3}(0,0)=\frac{1}{2\pi\sqrt{3}}.$$

Thus, we conclude

$$f_{Y_1|Y_2=0,Y_3=0}(y)=\frac{\left(\frac{1}{2\pi}\right)^{3/2}\frac{1}{\sqrt{4}}\,e^{-\frac{1}{2}\frac{3}{4}y^2}}{\frac{1}{2\pi\sqrt{3}}}=\frac{1}{\sqrt{2\pi}}\frac{\sqrt{3}}{2}\exp\left\{-\frac{1}{2}\left(\frac{y}{2/\sqrt{3}}\right)^2\right\}$$

which we recognize as the density function of a normal random variable with mean 0 and variance 3/4. That is,

$$Y_1|Y_2=Y_3=0\in N\left(0,\tfrac{3}{4}\right).$$