Stat 351 Fall 2006
Assignment #7 Solutions

- If X_1, X_2 are independent $N(0,1)$ random variables, then $Y_1 = X_1 - 3X_2 + 2$ is normal with mean $E(Y_1) = E(X_1) - 3E(X_2) + 2 = 2$ and variance $\operatorname{var}(Y_1) = \operatorname{var}(X_1 - 3X_2 + 2) = \operatorname{var}(X_1) + 9\operatorname{var}(X_2) - 6\operatorname{cov}(X_1, X_2) = 1 + 9 - 0 = 10$, and $Y_2 = 2X_1 - X_2 - 1$ is normal with mean $E(Y_2) = 2E(X_1) - E(X_2) - 1 = -1$ and variance $\operatorname{var}(Y_2) = \operatorname{var}(2X_1 - X_2 - 1) = 4\operatorname{var}(X_1) + \operatorname{var}(X_2) - 4\operatorname{cov}(X_1, X_2) = 4 + 1 - 0 = 5$. Since $\operatorname{cov}(Y_1, Y_2) = \operatorname{cov}(X_1 - 3X_2 + 2, 2X_1 - X_2 - 1) = 2\operatorname{var}(X_1) - 7\operatorname{cov}(X_1, X_2) + 3\operatorname{var}(X_2) = 2 - 0 + 3 = 5$, we conclude that $\mathbf{Y} = (Y_1, Y_2)'$ is multivariate normal $N(\boldsymbol{\mu}, \boldsymbol{\Lambda})$ where

$$\boldsymbol{\mu} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad \text{and} \quad \boldsymbol{\Lambda} = \begin{pmatrix} 10 & 5 \\ 5 & 5 \end{pmatrix}.$$

- Let

$$B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \end{pmatrix}$$

so that $\mathbf{Y} = B\mathbf{X}$. By Theorem 3.1, $\mathbf{Y}$ is MVN with mean

$$B\boldsymbol{\mu} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \\ -3 \end{pmatrix} = \begin{pmatrix} 0 \\ 8 \end{pmatrix}$$

and covariance matrix

$$B\boldsymbol{\Lambda}B' = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 & 3 \\ 1 & 4 & -2 \\ 3 & -2 & 8 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 16 & -2 \\ -2 & 16 \end{pmatrix}.$$

- Let

$$B = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & -2 \\ -2 & 0 & 3 \end{pmatrix}$$

so that $\mathbf{Y} = B\mathbf{X}$. By Theorem 3.1, $\mathbf{Y}$ is MVN with mean

$$B\boldsymbol{\mu} = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & -2 \\ -2 & 0 & 3 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

and covariance matrix

$$B\boldsymbol{\Lambda}B' = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & -2 \\ -2 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & -2 \\ 0 & 1 & 0 \\ -1 & -2 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 4 & -5 \\ 4 & 9 & -10 \\ -5 & -10 & 13 \end{pmatrix}.$$

- Let

$$B = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

so that $\mathbf{Y} = B\mathbf{X}$. By Theorem 3.1, $\mathbf{Y}$ is MVN with mean

$$B\boldsymbol{\mu} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

and covariance matrix

$$B\mathbf{\Lambda}B' = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

Exercise 4.2, page 127: If $\phi(t,u) = \exp\{it - 2t^2 - u^2 - tu\} = \exp\{it - \frac{1}{2}(4t^2 + 2tu + 2u^2)\}$ then we recognize this as the characteristic function of a normal random variable

$$\mathbf{X} = (X_1, X_2)' \in N\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 & 1 \\ 1 & 2 \end{pmatrix} \right).$$

Therefore, by Defintion I, $X_1 + X_2$ is normal with mean $E(X_1) + E(X_2) = 0 + 1 = 1$ and variance $\mathrm{var}(X_1 + X_2) = \mathrm{var}(X_1) + \mathrm{var}(X_2) + 2\,\mathrm{cov}(X_1, X_2) = 4 + 2 + 2 \cdot 1 = 8$. That is,

$$X_1 + X_2 \in N(1, 8).$$

Exercise 5.2, page 129: If $\psi(t,u) = \exp\{t^2 + 3tu + 4u^2\} = \exp\{\frac{1}{2}(2t^2 + 6tu + 8u^2)\}$ then we recognize this as the moment generating function of a normal random variable

$$\mathbf{X} = (X_1, X_2)' \in N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 & 3 \\ 3 & 8 \end{pmatrix} \right).$$

Since

$$\rho = \mathrm{corr}(X, Y) = \frac{\mathrm{cov}(X,Y)}{\sqrt{\mathrm{var}(X) \cdot \mathrm{var}(Y)}} = \frac{3}{\sqrt{2 \cdot 8}} = \frac{3}{4},$$

we conclude that the density function of $\mathbf{X}$ is given by

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{2\pi \cdot 2 \cdot 8 \cdot \sqrt{1 - (3/4)^2}} \exp\left\{ -\frac{1}{2(1 - (3/4)^2)} \left(\frac{x_1^2}{4} - 2\frac{3}{4}\frac{x_1 x_2}{2 \cdot 8} + \frac{x_2^2}{64} \right) \right\}.$$

Exercise 7.1, page 134: By Definition I, we know that X and $Y - \rho X$ are normally distributed. Therefore, by Theorem 7.1, X and $Y - \rho X$ are independent if and only if $\mathrm{cov}(X, Y - \rho X) = 0$. We compute

$$\begin{aligned} \mathrm{cov}(X, Y - \rho X) = \mathrm{cov}(X,Y) - \mathrm{cov}(X, \rho X) = \mathrm{cov}(X,Y) - \rho\,\mathrm{var}(X) &= \rho\,\mathrm{SD}(X)\,\mathrm{SD}(Y) - \rho\,\mathrm{var}(X) \\ &= \rho\,\mathrm{var}(X) - \rho\,\mathrm{var}(X) = 0 \end{aligned}$$

since $\mathrm{SD}(X) \cdot \mathrm{SD}(Y) = \mathrm{SD}(X) \cdot \mathrm{SD}(X) = \mathrm{var}(X)$ by the assumption that $\mathrm{var}(X) = \mathrm{var}(Y)$. Hence, X and $Y - \rho X$ are in fact independent.