Stat 351 Fall 2006
Assignment #6 Solutions (Part II)

Problem #7, page 115: By Theorem IV.3.1, the joint density

$$f_{X_{(1)},X_{(2)},X_{(3)},X_{(4)}}(y_1,y_2,y_3,y_4) = 24$$

provided that $0 < y_1 < y_2 < y_3 < y_4 < 1$. Therefore,

$$\begin{aligned} f_{X_{(3)},X_{(4)}}(y_3,y_4) = \int_0^{y_3}\int_0^{y_2} f_{X_{(1)},X_{(2)},X_{(3)},X_{(4)}}(y_1,y_2,y_3,y_4)\,dy_1\,dy_2 &= \int_0^{y_3}\int_0^{y_2} 24\,dy_1\,dy_2 \\ &= \int_0^{y_3} 24y_2\,dy_2 \\ &= 12y_3^2 \end{aligned}$$

provided that $0 < y_3 < y_4 < 1$. We then see that

$$P(X_{(3)} + X_{(4)} \le 1) = \int_0^{1/2}\int_y^{1-y} f_{X_{(3)},X_{(4)}}(y,z)\,dz\,dy$$

where $f_{X_{(3)},X_{(4)}}(y,z) = 12y^2$ for $0 < y < z < 1$. This then gives

$$P(X_{(3)} + X_{(4)} \le 1) = \int_0^{1/2}\int_y^{1-y} 12y^2\;dz\,dy = \int_0^{1/2} 12y^2(1-2y)\,dy = (4y^3 - 6y^4)\Big|_0^{1/2} = \frac{1}{8}.$$

Problem #8, page 115: By definition,

$$\rho_{X_{(1)},X_{(3)}} = \frac{\operatorname{cov}(X_{(1)},X_{(3)})}{\sqrt{\operatorname{var}(X_{(1)})\cdot\operatorname{var}(X_{(3)})}}.$$

Since X_1, X_2, X_3 are iid Exp(1) random variables, we conclude from Theorem IV.2.1 that

$$f_{X_{(1)},X_{(3)}}(y_1,y_3) = 6(e^{-y_1} - e^{-y_3})e^{-y_1}e^{-y_3}$$

provided $0 < y_1 < y_3 < \infty$. We also conclude from Theorem IV.1.2 (or, equivalently, page 103) that

$$f_{X_{(1)}}(y_1) = 3(e^{-y_1})^2 e^{-y_1} = 3e^{-3y_1}$$

provided that $0 < y_1 < \infty$, and that

$$f_{X_{(3)}}(y_3) = 3(1 - e^{-y_3})^2 e^{-y_3}$$

provided that $0 < y_3 < \infty$. Since we recognize $X_{(1)} \in \text{Exp}(1/3)$ we conclude immediately that $E(X_{(1)}) = 1/3$ and $\operatorname{var}(X_{(1)}) = 1/9$. Next we compute

$$\begin{aligned} E(X_{(3)}) = \int_0^\infty 3y_3(1-e^{-y_3})^2 e^{-y_3}\,dy_3 &= \int_0^\infty 3y_3e^{-y_3}\,dy_3 - \int_0^\infty 6y_3e^{-2y_3}\,dy_3 + \int_0^\infty 3y_3e^{-3y_3}\,dy_3 \\ &= 3\Gamma(2) - 6\left(\frac{1}{2}\right)^2\Gamma(2) + 3\left(\frac{1}{3}\right)^2\Gamma(2) \\ &= \frac{11}{6} \end{aligned}$$

and

$$E(X_{(3)}^2) = \int_0^\infty 3y_3^2(1-e^{-y_3})^2 e^{-y_3}\,dy_3 = \int_0^\infty 3y_3^2 e^{-y_3}\,dy_3 - \int_0^\infty 6y_3^2 e^{-2y_3}\,dy_3 + \int_0^\infty 3y_3^2 e^{-3y_3}\,dy_3$$
$$= 3\Gamma(3) - 6\left(\frac{1}{2}\right)^3\Gamma(3) + 3\left(\frac{1}{3}\right)^3\Gamma(3)$$
$$= \frac{85}{18}.$$

Therefore,

$$\text{var}(X_{(3)}) = E(X_{(3)}^2) - [E(X_{(3)})]^2 = \frac{85}{18} - \left(\frac{11}{6}\right)^2 = \frac{49}{36}.$$

Now we compute

$$\begin{aligned}
E(X_{(1)}X_{(3)}) &= \int_{-\infty}^\infty \int_{-\infty}^\infty f_{X_{(1)},X_{(3)}}(y_1,y_3)\,dy_3\,dy_1 = \int_0^\infty \int_{y_1}^\infty 6y_1y_3(e^{-y_1}-e^{-y_3})e^{-y_1}e^{-y_3}\,dy_3\,dy_1 \\
&= 6\int_0^\infty y_1e^{-y_1}\int_{y_1}^\infty y_3(e^{-y_1}-e^{-y_3})e^{-y_3}\,dy_3\,dy_1. \\
&= 6\int_0^\infty y_1e^{-2y_1}\int_{y_1}^\infty y_3e^{-y_3}\,dy_3\,dy_1 - 6\int_0^\infty y_1e^{-y_1}\int_{y_1}^\infty y_3e^{-2y_3}\,dy_3\,dy_1 \\
&= 6\int_0^\infty y_1e^{-2y_1}(y_1e^{-y_1}+e^{-y_1})\,dy_1 - 6\int_0^\infty y_1e^{-y_1}\left(\frac{1}{2}y_1e^{-2y_1}+\frac{1}{4}e^{-2y_1}\right)dy_1 \\
&= 3\int_0^\infty y_1^2e^{-3y_1}\,dy_1 + \frac{9}{2}\int_0^\infty y_1e^{-3y_1}\,dy_1 \\
&= 3\Gamma(3)\left(\frac{1}{3}\right)^3 + \frac{9}{2}\Gamma(2)\left(\frac{1}{3}\right)^2 \\
&= \frac{13}{18}
\end{aligned}$$

so that

$$\text{cov}(X_{(1)},X_{(3)}) = E(X_{(1)}X_{(3)}) - E(X_{(1)})E(X_{(3)}) = \frac{13}{18} - \frac{1}{3}\cdot\frac{11}{6} = \frac{1}{9}.$$

Finally, we put everything together so that

$$\rho_{X_{(1)},X_{(3)}} = \frac{\text{cov}(X_{(1)},X_{(3)})}{\sqrt{\text{var}(X_{(1)})\cdot\text{var}(X_{(3)})}} = \frac{1/9}{1/3\cdot 7/6} = \frac{2}{7}.$$