Stat 351 Fall 2006
Solutions to Selected Problems

Problem #11, page 56: Since

$$\int_0^1 \int_x^1 cx^2 \, dy \, dx = c \int_0^1 x^2(1-x) \, dx = c \left[\frac{1}{3}x^3 - \frac{1}{4}x^4 \right]_0^1 = \frac{c}{12}$$

we conclude that $c = 12$. The marginal for Y is therefore given by

$$f_Y(y) = \int_0^y 12x^2 \, dx = 4y^3, \quad 0 < y < 1$$

and the marginal for X is

$$f_X(x) = \int_x^1 12x^2 \, dy = 12x^2(1-x), \quad 0 < x < 1.$$

Hence we compute

$$E(Y) = \int_0^1 y \cdot 4y^3 \, dy = \frac{4}{5}$$

and

$$E(X) = \int_0^1 x \cdot 12x^2(1-x) \, dx = 3 - \frac{12}{5} = \frac{3}{5}.$$

The conditional densities are then

$$f_{X|Y=y}(x) = \frac{12x^2}{4y^3} = \frac{3x^2}{y^3}, \quad 0 < x < y$$

and

$$f_{Y|X=x}(y) = \frac{12x^2}{12x^2(1-x)} = \frac{1}{1-x}, \quad x < y < 1.$$

Finally, we find

$$E(X|Y=y) = \int_0^y x \cdot \frac{3x^2}{y^3} \, dx = \frac{3y}{4}$$

and

$$E(Y|X=x) = \int_x^1 y \cdot \frac{1}{1-x} \, dy = \frac{1-x^2}{2(1-x)} = \frac{1+x}{2}.$$

Problem #12, page 57: Since

$$\int_0^1 \int_0^x cx^2 y \, dy \, dx = \frac{c}{2} \int_0^1 x^4 \, dx = \frac{c}{2} \left[\frac{1}{5}x^5 \right]_0^1 = \frac{c}{10}$$

we conclude that $c = 10$. The marginal for Y is therefore given by

$$f_Y(y) = \int_y^1 10x^2 y \, dx = \frac{10}{3} y(1-y^3), \quad 0 < y < 1$$

and the marginal for X is

$$f_X(x) = \int_0^x 10x^2 y \, dy = 5x^4, \quad 0 < x < 1.$$

Hence we compute

$$E(Y) = \int_0^1 y \cdot \frac{10}{3} y(1-y^3)\, dy = \frac{10}{9} - \frac{10}{18} = \frac{5}{9}$$

and

$$E(X) = \int_0^1 x \cdot 5x^4\, dx = \frac{5}{6}.$$

The conditional densities are then

$$f_{X|Y=y}(x) = \frac{10x^2 y}{\frac{10}{3} y(1-y^3)} = \frac{3x^2}{1-y^3}, \quad y < x < 1$$

and

$$f_{Y|X=x}(y) = \frac{10x^2 y}{5x^4} = \frac{2y}{x^2}, \quad 0 < y < x.$$

Finally, we find

$$E(X|Y=y) = \int_y^1 x \cdot \frac{3x^2}{1-y^3}\, dx = \frac{3(1-y^4)}{4(1-y^3)}$$

and

$$E(Y|X=x) = \int_0^x y \cdot \frac{2y}{x^2}\, dy = \frac{2x}{3}.$$

Problem #13, page 57: Since

$$\int_0^1 \int_x^1 c(x+y)\, dy\, dx = c \int_0^1 \left[x(1-x) + \frac{1}{2}(1-x^2) \right] dx = c \left[\frac{1}{2}x + \frac{1}{2}x^2 - \frac{1}{2}x^3 \right]_0^1 = \frac{c}{2}$$

we conclude that $c = 2$. The marginal for Y is therefore given by

$$f_Y(y) = \int_0^y 2(x+y)\, dx = y^2 + 2y^2 = 3y^2, \quad 0 < y < 1$$

and the marginal for X is

$$f_X(x) = \int_x^1 2(x+y)\, dy = 2x(1-x) + (1-x^2) = 1 + 2x - 3x^2, \quad 0 < x < 1.$$

Hence we compute

$$E(Y) = \int_0^1 y \cdot 3y^2\, dy = \frac{3}{4}$$

and

$$E(X) = \int_0^1 x \cdot (1 + 2x - 3x^2)\, dx = \frac{1}{2} + \frac{2}{3} - \frac{3}{4} = \frac{5}{12}.$$

The conditional densities are then

$$f_{X|Y=y}(x) = \frac{2(x+y)}{3y^2}, \quad 0 < x < y$$

and

$$f_{Y|X=x}(y) = \frac{2(x+y)}{1 + 2x - 3x^2}, \quad x < y < 1.$$

Finally, we find

$$E(X|Y=y) = \int_0^y x \cdot \frac{2(x+y)}{3y^2}\, dx = \frac{2}{3y^2} \cdot \left(\frac{y^3}{3} + \frac{y^3}{2}\right) = \frac{5y}{9}$$

and

$$\begin{aligned} E(Y|X=x) = \int_x^1 y \cdot \frac{2(x+y)}{1+2x-3x^2}\, dy = \frac{2}{1+2x-3x^2} \cdot \left(\frac{x(1-x^2)}{2} + \frac{(1-x^3)}{3}\right) &= \frac{2+3x-5x^3}{3(1+2x-3x^2)} \\ &= \frac{(2+5x+5x^2)(1-x)}{3(3x+1)(1-x)} \\ &= \frac{2+5x+5x^2}{3(3x+1)}. \end{aligned}$$

Problem #14, page 57: Since

$$\int_0^1 \int_{x^2}^x c\, dy\, dx = c \int_0^1 (x - x^2)\, dx = c\left[\frac{1}{2}x^2 - \frac{1}{3}x^3\right]_0^1 = \frac{c}{6}$$

we conclude that $c = 6$. The marginal for Y is therefore given by

$$f_Y(y) = \int_y^{\sqrt{y}} 6\, dx = 6(\sqrt{y} - y), \quad 0 \le y \le 1$$

and the marginal for X is

$$f_X(x) = \int_{x^2}^x 6\, dy = 6(x - x^2) = 6x(1-x), \quad 0 \le x \le 1.$$

The conditional densities are then

$$f_{X|Y=y}(x) = \frac{6}{6(\sqrt{y}-y)} = \frac{1}{\sqrt{y}-y}, \quad y \le x \le \sqrt{y}$$

and

$$f_{Y|X=x}(y) = \frac{6}{6x(1-x)} = \frac{1}{x(1-x)}, \quad x^2 \le y \le x.$$

Finally, we find

$$E(X|Y=y) = \int_y^{\sqrt{y}} x \cdot \frac{1}{\sqrt{y}-y}\, dx = \frac{y - y^2}{2(\sqrt{y}-y)} = \frac{y+\sqrt{y}}{2}$$

and

$$E(Y|X=x) = \int_{x^2}^x y \cdot \frac{1}{x(1-x)}\, dy = \frac{x^2 - x^4}{2x(1-x)} = \frac{x(1+x)}{2}.$$

Problem #15, page 57: Since

$$\int_0^1 \int_0^{\sqrt{1-x^2}} cx^3 y\, dy\, dx = \frac{c}{2}\int_0^1 x^3(1-x^2)\, dx = \frac{c}{2}\left[\frac{1}{4}x^4 - \frac{1}{6}x^6\right]_0^1 = \frac{c}{24}$$

we conclude that $c = 24$. The marginal for Y is therefore given by

$$f_Y(y) = \int_0^{\sqrt{1-y^2}} 24x^3 y\, dx = 6y(1-y^2)^2, \quad 0 < y \le 1$$

and the marginal for X is

$$f_X(x) = \int_0^{\sqrt{1-x^2}} 24x^3 y\, dy = 12x^3(1-x^2), \quad 0 < x \le 1.$$

The conditional densities are then

$$f_{X|Y=y}(x) = \frac{24x^3 y}{6y(1-y^2)^2} = \frac{4x^3}{(1-y^2)^2}, \quad 0 < x \le \sqrt{1-y^2}$$

and

$$f_{Y|X=x}(y) = \frac{24x^3 y}{12x^3(1-x^2)} = \frac{2y}{1-x^2}, \quad 0 < y \le \sqrt{1-x^2}.$$

Finally, we find

$$E(X|Y=y) = \int_0^{\sqrt{1-y^2}} x \cdot \frac{4x^3}{(1-y^2)^2}\, dx = \frac{4(1-y^2)^{5/2}}{5(1-y^2)^2} = \frac{4\sqrt{1-y^2}}{5}$$

and

$$E(Y|X=x) = \int_0^{\sqrt{1-x^2}} y \cdot \frac{2y}{1-x^2}\, dy = \frac{2(1-x^2)^{3/2}}{3(1-x^2)} = \frac{2\sqrt{1-x^2}}{3}.$$

Problem #16, page 57: Since

$$\int_0^{1/2}\int_0^{\sqrt{1-4x^2}} cxy\, dy\, dx = \frac{c}{2}\int_0^{1/2} x(1-4x^2)\, dx = \frac{c}{2}\left[\frac{1}{2}x^2 - x^4\right]_0^{1/2} = \frac{c}{32}$$

we conclude that $c = 32$. The marginal for Y is therefore given by

$$f_Y(y) = \int_0^{\frac{1}{2}\sqrt{1-y^2}} 32xy\, dx = 16y \cdot \frac{1}{4}(1-y^2) = 4y(1-y^2), \quad 0 < y \le \frac{1}{2}$$

and the marginal for X is

$$f_X(x) = \int_0^{\sqrt{1-4x^2}} 32xy\, dy = 16x(1-4x^2), \quad 0 < x \le \frac{1}{2}.$$

The conditional densities are then

$$f_{X|Y=y}(x) = \frac{32xy}{4y(1-y^2)} = \frac{8x}{1-y^2}, \quad 0 < x \le \frac{1}{2}\sqrt{1-y^2}$$

and

$$f_{Y|X=x}(y) = \frac{32xy}{16x(1-4x^2)} = \frac{2y}{1-4x^2}, \quad 0 < y \le \sqrt{1-4x^2}.$$

Finally, we find

$$E(X|Y=y) = \int_0^{\frac{1}{2}\sqrt{1-y^2}} x \cdot \frac{8x}{1-y^2}\, dx = \frac{8 \cdot \frac{1}{8} \cdot (1-y^2)^{3/2}}{3(1-y^2)} = \frac{\sqrt{1-y^2}}{3}$$

and

$$E(Y|X=x) = \int_0^{\sqrt{1-4x^2}} y \cdot \frac{2y}{1-4x^2}\, dy = \frac{2(1-4x^2)^{3/2})}{3(1-4x^2)} = \frac{2\sqrt{1-4x^2}}{3}.$$