Stat 351 Fall 2006
Solutions to Assignment #5

**Problem #13, page 28**: Suppose that $Y \in \chi^2(n)$ so that the density of $Y$ is given by

$$f_Y(y) = \frac{1}{\Gamma(\frac{n}{2})} \cdot y^{n/2-1} \cdot 2^{-n/2} \cdot e^{-y/2}, \quad 0 < y < \infty.$$

Let $U = \frac{1}{\sqrt{Y}}$. Since $y > 0$, the distribution function of $U$ is given by

$$F_U(u) = P(U \le u) = P\left(\frac{1}{\sqrt{Y}} \le u\right) = P(Y \ge u^{-2}) = \int_{u^{-2}}^{\infty} f_Y(y)\, dy = -\int_{\infty}^{u^{-2}} f_Y(y)\, dy.$$

Taking derivatives with respect to $u$ gives

$$f_U(u) = 2u^{-3} f_Y(u^{-2}) = 2u^{-3} \cdot \frac{1}{\Gamma(\frac{n}{2})} \cdot u^{2-n} \cdot 2^{-n/2} \cdot e^{-u^{-2}/2} = \frac{1}{\Gamma(\frac{n}{2})} \cdot u^{-n-1} \cdot 2^{1-n/2} \cdot e^{-1/(2u^2)}$$

for $0 < u < \infty$. The mean of $U$ is then given by

$$E(U) = \int_0^\infty u \cdot f_U(u)\, du = \int_0^\infty u \cdot \frac{1}{\Gamma(\frac{n}{2})} \cdot u^{-n-1} \cdot 2^{1-n/2} \cdot e^{-1/(2u^2)}\, du = \frac{1}{\Gamma(\frac{n}{2})} \cdot 2^{1-n/2} \int_0^\infty u^{-n} \cdot e^{-1/(2u^2)}\, du.$$

In order to evaluate this integral, we make the substitution $v = u^{-2}/2$ so that $dv = -u^{-3} du$. Therefore,

$$\int_0^\infty u^{-n} \cdot e^{-1/(2u^2)}\, du = \int_0^\infty (2v)^{n/2-3/2} \cdot e^{-v}\, dv = 2^{(n-3)/2} \int_0^\infty v^{(n-1)/2-1} \cdot e^{-v}\, dv = 2^{(n-3)/2} \cdot \Gamma(\tfrac{n-1}{2}).$$

That is,

$$E\left(\frac{1}{\sqrt{Y}}\right) = E(U) = \frac{1}{\Gamma(\frac{n}{2})} \cdot 2^{1-n/2} \cdot 2^{(n-3)/2} \cdot \Gamma\left(\tfrac{n-1}{2}\right) = \frac{\Gamma\left(\frac{n-1}{2}\right)}{\sqrt{2} \cdot \Gamma(\frac{n}{2})}.$$

**Problem #23, page 29**: Suppose that $X$ and $Y$ have joint density

$$f_{X,Y}(x,y) = \begin{cases} \frac{x}{(1+x)^2(1+xy)^2}, & \text{for } x, y > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Let $U = X$ and $V = XY$ so that solving for $X$ and $Y$ gives

$$X = U \quad \text{and} \quad Y = V/U.$$

The Jacobian of this transformation is given by

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -v/u^2 & 1/u \end{vmatrix} = \frac{1}{u}.$$

The density of $(U,V)$ is therefore given by

$$f_{U,V}(u,v) = f_{X,Y}(u, v/u) \cdot |J| = \frac{u}{(1+u)^2(1+u \cdot v/u)^2} \cdot \frac{1}{u} = \frac{1}{(1+u)^2} \cdot \frac{1}{(1+v)^2},$$

provided that $0 < u < \infty$, $0 < v < \infty$. Since we can write the joint density as a product of a function of $u$ only multiplied by a function of $v$ only, we conclude that $U$ and $V$ are independent. That is,

$$f_{U,V}(u,v) = f_U(u) \cdot f_V(v)$$

where

$$f_U(u) = \frac{1}{(1+u)^2} \text{ for } u > 0, \quad \text{and} \quad f_V(v) = \frac{1}{(1+v)^2} \text{ for } v > 0.$$

Notice that both $U$ and $V$ have the same distribution, namely $F(2,2)$. (See page 261.)

**Problem #24, page 29**: Suppose that $X$ and $Y$ have joint density

$$f_{X,Y}(x,y) = \begin{cases} \frac{2}{(1+x+y)^3} & \text{for } x, y > 0, \\ 0, & \text{otherwise.} \end{cases}$$

**(a)** Let $U = X + Y$ and $V = \frac{X}{X+Y}$, so that solving for $X$ and $Y$ gives

$$X = UV \quad \text{and} \quad Y = U - UV.$$

The Jacobian of this transformation is given by

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ 1-v & -u \end{vmatrix} = -u.$$

The density of $(U, V)$ is therefore given by

$$f_{U,V}(u,v) = f_{X,Y}(uv, u-uv) \cdot |J| = \frac{2}{(1+uv+u-uv)^3} \cdot u = \frac{2u}{(1+u)^3},$$

provided that $0 < u < \infty$, $0 < v < 1$. Since we can write the joint density as a product of a function of $u$ only multiplied by a function of $v$ only, we conclude that

$$f_{U,V}(u,v) = f_U(u) \cdot f_V(v)$$

where

$$f_U(u) = \frac{2u}{(1+u)^3} \text{ for } u > 0, \quad \text{and} \quad f_V(v) = 1 \text{ for } 0 < v < 1.$$

Therefore, the density of $X + Y$ is

$$f_{X+Y}(u) = \frac{2u}{(1+u)^3} \text{ for } u > 0.$$

**(b)** Let $U = X - Y$ and $V = X$, so that solving for $X$ and $Y$ gives

$$X = V \quad \text{and} \quad Y = V - U.$$

The Jacobian of this transformation is given by

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ -1 & 1 \end{vmatrix} = 1.$$

The density of $(U,V)$ is therefore given by

$$f_{U,V}(u,v) = f_{X,Y}(v, v-u) \cdot |J| = \frac{2}{(1+v+v-u)^3} \cdot 1 = \frac{2}{(1+2v-u)^3},$$

provided that $v > u$ and $v > 0$ (i.e., $v > \max\{0,u\}$), and $-\infty < u < \infty$. If $u > 0$, then $\max\{u,0\} = u$, and so we calculate

$$f_U(u) = \int_{-\infty}^{\infty} f_{U,V}(u,v)\, dv = \int_u^{\infty} \frac{2}{(1+2v-u)^3}\, dv = \frac{1}{2(1+2v-u)^2}\Bigg|_u^{\infty} = \frac{1}{2(1+u)^2}.$$

If $u \le 0$, then $\max\{u,0\} = 0$, and so we calculate

$$f_U(u) = \int_{-\infty}^{\infty} f_{U,V}(u,v)\, dv = \int_0^{\infty} \frac{2}{(1+2v-u)^3}\, dv = \frac{1}{2(1+2v-u)^2}\Bigg|_0^{\infty} = \frac{1}{2(1-u)^2}.$$

Therefore, the density of $X+Y$ is

$$f_{X+Y}(u) = \frac{1}{2(1+|u|)^2} \quad \text{for } -\infty < u < \infty.$$

**Problem #26, page 30**: Suppose that $X$ and $Y$ have joint density

$$f_{X,Y}(x,y) = \begin{cases} \lambda^2 e^{-\lambda y}, & \text{for } 0 < x < y, \\ 0, & \text{otherwise.} \end{cases}$$

Let $U = Y$ and $V = \frac{X}{Y-X}$, so that solving for $X$ and $Y$ gives

$$X = \frac{UV}{1+V} \quad \text{and} \quad Y = U.$$

The Jacobian of this transformation is given by

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v(1+v)^{-1} & u(1+v)^{-2} \\ 1 & 0 \end{vmatrix} = -\frac{u}{(1+v)^2}.$$

The density of $(U,V)$ is therefore given by

$$f_{U,V}(u,v) = f_{X,Y}(uv(1+v)^{-1}, u) \cdot |J| = \lambda^2 e^{-\lambda u} \cdot \frac{u}{(1+v)^2} = \lambda^2 u\, e^{-\lambda u} \cdot \frac{1}{(1+v)^2},$$

provided that $0 < u < \infty$, $0 < v < \infty$. Since we can write the joint density as a product of a function of $u$ only multiplied by a function of $v$ only, we conclude that $U$ and $V$ are independent. That is,

$$f_{U,V}(u,v) = f_U(u) \cdot f_V(v)$$

where

$$f_U(u) = \lambda^2 u\, e^{-\lambda u} \text{ for } u > 0, \quad \text{and} \quad f_V(v) = \frac{1}{(1+v)^2} \text{ for } v > 0.$$

Notice that $U \in \Gamma(2, \lambda^{-1})$ and that $V \in F(2,2)$. (See pages 260–261.)