Stat 351 Fall 2006
Solutions to Selected Problems

Problem #3, page 27: Suppose that $T \in t(n)$ so that the density of T is given by

$$f_T(x) = \frac{\Gamma(\frac{n+1}{2})}{\sqrt{\pi n}\,\Gamma(\frac{n}{2})} \cdot \left(1 + \frac{x^2}{n}\right)^{-(n+1)/2}, \quad -\infty < x < \infty.$$

Let $Y = T^2$. If $y \geq 0$, then the distribution function of Y is given by

$$\begin{aligned} F_Y(y) = P(Y \leq y) = P(T^2 \leq y) = P(-\sqrt{y} \leq T \leq \sqrt{y}) &= \int_{-\sqrt{y}}^{\sqrt{y}} f_T(x)\,dx \\ &= \int_0^{\sqrt{y}} f_T(x)\,dx - \int_0^{-\sqrt{y}} f_T(x)\,dx. \end{aligned}$$

Taking derivatives with respect to y gives

$$\begin{aligned} f_Y(y) = f_T(\sqrt{y}) \cdot \frac{1}{2\sqrt{y}} - f_T(-\sqrt{y}) \cdot \frac{-1}{2\sqrt{y}} &= \frac{1}{2\sqrt{y}}\,(\,f_T(\sqrt{y}) + f_T(-\sqrt{y})\,) \\ &= \frac{\Gamma(\frac{n+1}{2})}{\sqrt{\pi n y}\,\Gamma(\frac{n}{2})} \cdot \left(1 + \frac{y}{n}\right)^{-(n+1)/2} \\ &= \frac{\Gamma(\frac{1+n}{2})\left(\frac{1}{n}\right)^{1/2}}{\Gamma(\frac{1}{2})\,\Gamma(\frac{n}{2})} \cdot \frac{y^{1/2-1}}{(1+\frac{y}{n})^{(1+n)/2}}, \quad y \geq 0. \end{aligned}$$

In order to write this last line, we have used the fact that $\Gamma(1/2) = \sqrt{\pi}$. Notice that this is the density of an $F(1, n)$ random variable. (See page 261.)

Problem #6, page 27: If $X \in \beta(1, 1)$, then the density function of X is

$$f_X(x) = \frac{\Gamma(1+1)}{\Gamma(1)\Gamma(1)} x^{1-1}(1-x)^{1-1} = 1, \quad 0 < x < 1.$$

(We have used the fact that $\Gamma(2) = \Gamma(1) = 1$.) Since the density of X is also that of a uniform random variable, we conclude $X \in U(0, 1)$. Therefore, $\beta(1, 1) = U(0, 1)$.