Stat 351 Fall 2006
Assignment #3 Solutions

Problem #5, page 27: Suppose that $X \in C(0,1)$ so that the density of X is given by

$$f_X(x) = \frac{1}{\pi} \cdot \frac{1}{1+x^2}, \quad -\infty < x < \infty.$$

Let $Y = X^2$. If $y \geq 0$, then the distribution function of Y is given by

$$\begin{aligned} F_Y(y) = P(Y \leq y) = P(X^2 \leq y) = P(-\sqrt{y} \leq X \leq \sqrt{y}) &= \int_{-\sqrt{y}}^{\sqrt{y}} f_X(x)\,dx \\ &= \int_0^{\sqrt{y}} f_X(x)\,dx - \int_0^{-\sqrt{y}} f_X(x)\,dx. \end{aligned}$$

Taking derivatives with respect to y gives

$$f_Y(y) = f_X(\sqrt{y}) \cdot \frac{1}{2\sqrt{y}} - f_X(-\sqrt{y}) \cdot \frac{-1}{2\sqrt{y}} = \frac{1}{2\sqrt{y}}\left(f_X(\sqrt{y}) + f_X(-\sqrt{y}) \right) = \frac{1}{\pi\sqrt{y}} \cdot \frac{1}{1+y}, \quad y \geq 0.$$

Notice that this is the density of an $F(1,1)$ random variable. (See page 261 and recall that $\Gamma(1) = 1$, $\Gamma(1/2) = \sqrt{\pi}$.)

Problem #25, page 30: Suppose that $U = X^2 Y$ and let $V = X$. Solving for X and Y gives

$$X = V \quad \text{and} \quad Y = \frac{U}{V^2}.$$

The Jacobian of this transformation is given by

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ v^{-2} & -2uv^{-3} \end{vmatrix} = -v^{-2}.$$

If the density of (X,Y) is

$$f_{X,Y}(x,y) = \begin{cases} e^{-x^2 y}, & \text{for } x \geq 1,\ y > 0, \\ 0, & \text{otherwise,} \end{cases}$$

then the density of (U,V) is therefore given by

$$f_{U,V}(u,v) = f_{X,Y}(v, uv^{-2}) \cdot |J| = \frac{1}{v^2}\,e^{-u}$$

provided that $v \geq 1$ and $u > 0$. We can now determine the density of U as follows.

Routine Way: The marginal density of U is

$$f_U(u) = \int_{-\infty}^{\infty} f_{U,V}(u,v)\,dv = \int_1^{\infty} \frac{1}{v^2}\,e^{-u}\,dv = e^{-u}\left[-v^{-1}\right]_1^{\infty} = e^{-u}$$

for $u > 0$. We recognize that this is the density of an exponential random variable with parameter 1; that is, $U = X^2 Y \in \text{Exp}(1)$.

(continued)

Slick Way: Since the joint density of (U, V) is

$$f_{U,V}(u, v) = \begin{cases} v^{-2}e^{-u}, & \text{for } v \geq 1,\ u > 0, \\ 0, & \text{otherwise,} \end{cases}$$

we can immediately conclude that U and V are independent random variables with $f_V(v) = v^{-2}$ for $v \geq 1$ and $f_U(u) = e^{-u}$ for $u > 0$. And so we find (as before) that $U = X^2Y \in \text{Exp}(1)$.