Solutions to Math 312 Midterm Exam – October 11, 2013

1 (a). Observe that $w^3 = (\sqrt{3}+i)^3 = (\sqrt{3})^3 + 3(\sqrt{3})^2 i + 3i^2\sqrt{3} + i^3 = 8i$ and so

$$\frac{z}{w^3} = \frac{1-i}{8i} = -\frac{1}{8} - \frac{i}{8}.$$

That is, $a = b = -1/8$.

1 (b). Observe that the point $(-1/8, -1/8) \in \mathbb{R}^2$ lies in the third quadrant and makes an angle of $5\pi/4$ with the positive x-axis. Therefore, we can write

$$\frac{z}{w^3} = -\frac{1}{8} - \frac{i}{8} = \frac{1}{8}e^{i5\pi/4} = \frac{1}{8}e^{-i3\pi/4}$$

so that $\operatorname{Arg}(z/w^3) = -3\pi/4$.

2. Consider the equation $\zeta^4 = -i$. Writing $-i = e^{3i\pi/2}$ implies that the four values of ζ satisfying $\zeta^4 = -i$ are $\zeta_1 = e^{3i\pi/8}$, $\zeta_2 = e^{7i\pi/8}$, $\zeta_3 = e^{11i\pi/8}$, $\zeta_4 = e^{15i\pi/8}$. Since $2^4 = 16$, we conclude that the four values of z satisfying $z^4 = -16i$ are

$$z_1 = 2e^{3i\pi/8} = \sqrt{2-\sqrt{2}} + i\sqrt{2+\sqrt{2}}, \quad z_2 = 2e^{7i\pi/8} = -\sqrt{2+\sqrt{2}} + i\sqrt{2-\sqrt{2}},$$

$$z_3 = 2e^{11i\pi/8} = -\sqrt{2-\sqrt{2}} - i\sqrt{2+\sqrt{2}}, \quad z_4 = 2e^{15i\pi/8} = \sqrt{2+\sqrt{2}} - i\sqrt{2-\sqrt{2}}.$$

3. Observe that

$$f(z) = \frac{2+z}{1+z} = \frac{1+1+z}{1+z} = 1 + \frac{1}{1+z} = h_3 \circ h_2 \circ h_1(z)$$

where $h_1(z) = h_3(z) = z+1$ and $h_2(z) = 1/z$. Let $D_1 = h_1(D)$ so that $D_1 = \{z \in \mathbb{C} : |z-1| < 1\}$. Let $D_2 = h_2(D_1)$. In order to determine D_2, suppose that $z \in D_1$ and $w = 1/z = u + iv$. Hence,

$$|z-1| < 1 \iff |1/w - 1| < 1 \iff |1-w| < |w| \iff (u-1)^2 + v^2 < u^2 + v^2 \iff u > 1/2$$

and so $D_2 = \{z \in \mathbb{C} : \operatorname{Re}(z) > 1/2\}$. Finally, let $D_3 = h_3(D_2) = f(D)$ so that

$$f(D) = \{z \in \mathbb{C} : \operatorname{Re}(z) > 3/2\}.$$

4. Assume that $f(z)$ is analytic with $\operatorname{Re} f(z) = u(z)$. Since $f(z) = u(z) + iv(z)$ is assumed to be analytic, $u(z)$ and $v(z)$ satisfy the Cauchy-Riemann equations at all $z_0 = x_0 + iy_0 \in \mathbb{C}$. Since $u_x(z_0) = u_x(x_0, y_0) = 1 + e^{x_0}\sin y_0$ and $u_y(z_0) = u_y(x_0, y_0) = e^{x_0}\cos y_0$, the partial derivatives of $v(z) = v(x, y)$ satisfy

$$v_y(z_0) = v_y(x_0, y_0) = 1 + e^{x_0}\sin y_0 \quad \text{and} \quad v_x(z_0) = v_x(x_0, y_0) = -e^{x_0}\cos y_0.$$

Integrating the first expression implies $v(x_0, y_0) = y_0 - e^{x_0}\cos y_0 + c_1(x_0)$, whereas integrating the second expression implies $v(x_0, y_0) = -e^{x_0}\cos y_0 + c_2(y_0)$ where $c_1 = c_1(x)$ is a function of x only and $c_2 = c_2(y)$ is a function of y only. Comparing these two expressions implies that $v(x_0, y_0) = y_0 - e^{x_0}\cos y_0 + c$ for some real constant c. Thus, $f(z)$ is of the form $f(z) = u(z) + iv(z) = x + e^x \sin y + i(y - e^x\cos y + c)$. The assumption that $f(0) = i$ implies

$$f(z) = x + e^x\sin y + i(y - e^x\cos y + 2) = z - ie^z + 2i.$$

5. Since

$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\frac{\Delta z}{\overline{\Delta z} + i} - 0}{\Delta z} = \lim_{\Delta z \to 0} \frac{1}{\overline{\Delta z} + i} = \frac{1}{i} = -i$$

we conclude that $f(z)$ is differentiable at $z_0 = 0$ with $f'(0) = -i$.