Math 312 Midterm Exam – October 11, 2013

This exam is worth 50 points.

This exam has 5 problems and 1 numbered page.

You have 50 minutes to complete this exam. Please read all instructions carefully, and check your answers. Write your solutions in the exam booklets. Show all work neatly and in order, and clearly indicate your final answers. Answers must be justified whenever possible in order to earn full credit. ***Unless otherwise specified, no credit will be given for unsupported answers, even if your final answer is correct.*** *Points will be deducted for incoherent, incorrect, and/or irrelevant statements.*

This exam is closed-book, except that one $8\frac{1}{2} \times 11$ *double-sided page of handwritten notes is permitted. No other aids are allowed.*

Instructor: Michael Kozdron

1. (*8 points*)

Consider the complex variables $z = 1 - i$ and $w = \sqrt{3} + i$.

(i) Determine the values of $a \in \mathbb{R}$ and $b \in \mathbb{R}$ such that $\dfrac{z}{w^3} = a + ib$.

(ii) Determine $\operatorname{Arg}\left(\dfrac{z}{w^3}\right)$, the principal value of the argument of $\dfrac{z}{w^3}$.

2. (*8 points*) Find all complex variables z that satisfy $z^4 = -16i$. You may express your answer in either cartesian coordinates or polar form, depending on which you think is most convenient.

3. (*12 points*) Let $D = \{z \in \mathbb{C} : |z| < 1\}$ denote the open disk of radius 1 centred at the origin in the complex plane, and consider the function $f : D \to \mathbb{C}$ given by

$$f(z) = \frac{2+z}{1+z}.$$

Determine $f(D)$, the image of D under f. Express your answer both (a) analytically, and (b) graphically. Note that a simple sketch highlighting the key features of $f(D)$ will suffice for (b).

4. (*12 points*) Let $u : \mathbb{C} \to \mathbb{R}$ be given by $u(z) = u(x, y) = x + e^x \sin y$. Determine the unique analytic function $f : \mathbb{C} \to \mathbb{C}$ satisfying $\operatorname{Re} f(z) = u(z)$ and $f(0) = i$.

5. (*10 points*) Let $D = \{z \in \mathbb{C} : |z| < 1\}$ denote the open disk of radius 1 centred at the origin in the complex plane, and consider the function $f : D \to \mathbb{C}$ given by

$$f(z) = \frac{z}{\overline{z} + i}$$

Prove that $f(z)$ is differentiable at $z_0 = 0$ and compute $f'(0)$.