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Lecture #5: The Complex Exponential Function

Recall that last class we discussed the argument of a complex variable as well as some of the motivation for its definition.

Definition. Suppose that $z = x + iy \in \mathbb{C}$, $z \neq 0$. Define the *argument* of z, denoted $\arg z$, to be *any* solution θ of the pair of equations

$$\cos\theta = \frac{x}{|z|} \quad \text{and} \quad \sin\theta = \frac{y}{|z|},$$

and define the *principal value of the argument* of z, denoted $\operatorname{Arg} z$, to be the unique value of $\arg z \in (-\pi, \pi]$. If $z = 0$, we set $\arg 0 = \{0, \pm 2\pi, \pm 4\pi, \ldots\}$ so that $\operatorname{Arg} 0 = 0$.

Definition. Suppose that $z \in \mathbb{C}$. We define the *polar form* of z to be $re^{i\theta}$ where $r = |z|$ and $\theta = \operatorname{Arg} z$. For convenience, we will write $z = re^{i\theta}$.

Example 5.1. Write $z = -1 - i$ in polar form and identify $\arg z$.

Solution. If $z = -1 - i$, then $|z| = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2} = r$. Moreover,

$$\cos\theta = -\frac{1}{\sqrt{2}} \quad \text{and} \quad \sin\theta = -\frac{1}{\sqrt{2}}$$

implies that

$$\theta = \frac{5\pi}{4} + 2\pi k$$

for $k \in \mathbb{Z}$. Thus, $\operatorname{Arg} z = -3\pi/4$ and

$$\arg z = \left\{-\frac{3\pi}{4}, -\frac{3\pi}{4} \pm 2\pi, -\frac{3\pi}{4} \pm 4\pi, \ldots\right\} = \left\{-\frac{3\pi}{4} + 2\pi k : k \in \mathbb{Z}\right\} = \left\{\frac{5\pi}{4} + 2\pi k : k \in \mathbb{Z}\right\}.$$

Hence, the polar form of $z = -1 - i$ is $\sqrt{2}e^{-3i\pi/4}$. Equivalently, we can represent z as an ordered pair $(x, y) \in \mathbb{R}^2$ as

$$(-1, -1) = \left(\sqrt{2}\cos(-3\pi/4), \sqrt{2}\sin(-3\pi/4)\right).$$

Suppose that $z = re^{i\theta}$ is the polar form of $z \in \mathbb{C}$. As in the previous example, we can write z in cartesian coordinates as $(r\cos\theta, r\sin\theta)$. Using our identification of $(x, y) \in \mathbb{R}^2$ with $z = x + iy \in \mathbb{C}$, we conclude that an equivalent representation of z is

$$z = r\cos\theta + ir\sin\theta.$$

This is also sometimes called the polar form of z.

Definition. Suppose that $z \in \mathbb{C}$. The *polar form* of z is defined as

$$z = r\cos\theta + ir\sin\theta = re^{i\theta}$$

where $r = |z|$ and $\theta = \operatorname{Arg} z$.

If we take $r = 1$ in the definition of polar form, then we conclude that

$$\cos\theta + i\sin\theta = e^{i\theta}$$

which leads to the following definition.

Definition. The *complex exponential* $e^{i\theta}$ is defined as $e^{i\theta} = \cos\theta + i\sin\theta$.

Remark. If we take $\theta = \pi$ in definition of complex exponential, then we have one of the most magical formulas in all of mathematics:

$$e^{i\pi} = \cos(\pi) + i\sin(\pi) = -1 + i0 = -1,$$

or equivalently,

$$\boxed{e^{i\pi} + 1 = 0.}$$

This is **Euler's formula** relating all five fundamental constants of mathematics!!!! The constant e comes from calculus, π comes from geometry, i comes from algebra, and 1 is the basic unit for generating the arithmetic system from the usual counting numbers.

Properties of the Complex Exponential $e^{i\theta}$

Proposition 5.2. $e^{-i\theta} = \overline{e^{i\theta}}$

Proof. We find

$$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta) = \cos(\theta) - i\sin(\theta) = \overline{e^{i\theta}}$$

and the proof is complete. □

Proposition 5.3. $|e^{i\theta}| = 1$

Proof. Using the previous proposition, we find

$$|e^{i\theta}| = e^{i\theta}\overline{e^{i\theta}} = e^{i\theta}e^{-i\theta} = (\cos(\theta) + i\sin(\theta))(\cos(\theta) - i\sin(\theta)) = \cos^2(\theta) + \sin^2(\theta) = 1$$

as required. □

Proposition 5.4. $\dfrac{1}{e^{i\theta}} = e^{-i\theta}$

Proof. We find

$$\frac{1}{e^{i\theta}} = \frac{1}{\cos(\theta) + i\sin(\theta)} = \frac{1}{\cos(\theta) + i\sin(\theta)} \frac{\cos(\theta) - i\sin(\theta)}{\cos(\theta) - i\sin(\theta)} = \frac{\cos(\theta) - i\sin(\theta)}{|e^{i\theta}|}$$
$$= \cos(\theta) - i\sin(\theta)$$
$$= e^{-i\theta}$$

and the proof is complete. □

Proposition 5.5. $e^{i\theta} = e^{i(\theta+2\pi k)}$, $k \in \mathbb{Z}$

Proof. Since the real-valued sine and cosine functions are each 2π-periodic, we know that

$$\cos(\theta) = \cos(\theta + 2\pi k) \quad \text{and} \quad \sin(\theta) = \sin(\theta + 2\pi k)$$

so that

$$e^{i\theta} = \cos(\theta) + i\sin(\theta) = \cos(\theta + 2\pi k) + i\sin(\theta + 2\pi k) = e^{i(\theta+2\pi k)}$$

as required. □

Proposition 5.6. $e^{i\theta_1}e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$

Proof. By definition,

$$\begin{aligned} e^{i\theta_1}e^{i\theta_2} &= (\cos(\theta_1) + i\sin(\theta_1))(\cos(\theta_2) + i\sin(\theta_2)) \\ &= \cos(\theta_1)\cos(\theta_2) + i\cos(\theta_1)\sin(\theta_2) + i\sin(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2) \\ &= \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2) + i(\cos(\theta_1)\sin(\theta_2) + \sin(\theta_1)\cos(\theta_2)) \\ &= \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2) \\ &= e^{i(\theta_1+\theta_2)} \end{aligned}$$

completing the proof. □

Proposition 5.7. $\dfrac{e^{i\theta_1}}{e^{i\theta_2}} = e^{i(\theta_1-\theta_2)}$

Proof. Using our previous propositions, we find

$$\frac{e^{i\theta_1}}{e^{i\theta_2}} = e^{i\theta_1}e^{-i\theta_2} = e^{i\theta_1 - i\theta_2} = e^{i(\theta_1-\theta_2)}$$

as required. □

Corollary 5.8. *If $z_1 = r_1e^{i\theta_1}$ and $z_2 = r_2e^{i\theta_2}$, then*

$$z_1z_2 = r_1r_2e^{i(\theta_1+\theta_2)},$$

and if $z_2 \neq 0$, then

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}e^{i(\theta_1-\theta_2)}.$$

Exercise 5.9. Prove the previous corollary.

Powers: An Application of Complex Exponentials

Recall that if $a \in \mathbb{R}$ and n, $m \in \mathbb{Z}$, then $(a^n)^m = a^{nm}$. In particular, if $x \in \mathbb{R}$, then $(e^x)^n = e^{nx}$. As we will now show, this same sort of result is true for the complex exponential.

Theorem 5.10. *Let $z = re^{i\theta}$ be the polar form of the complex variable z. If n is a non-negative integer, then*

$$z^n = r^n e^{in\theta}.$$

Proof. The proof is by induction. Clearly it is true for $n = 1$. If $n = 2$, then we find from Corollary 5.8 that

$$z^2 = (re^{i\theta})(re^{i\theta}) = r^2 e^{i(\theta+\theta)} = r^2 e^{i2\theta}.$$

If $n = 3$, then

$$z^3 = z^2 z = (r^2 e^{i2\theta})(re^{i\theta}) = r^3 e^{i(2\theta+\theta)} = r^3 e^{i3\theta}.$$

In general, if $z^k = r^k e^{ik\theta}$ for some k, then

$$z^{k+1} = z^k z = (r^k e^{ik\theta})(re^{i\theta}) = r^{k+1} e^{i(k\theta+\theta)} = r^{k+1} e^{i(k+1)\theta}$$

which completes the proof. □

Note that this theorem can sometimes be used to simplify multiplication of complex variables.

Example 5.11. Determine the real and imaginary parts of $(-1 - i)^{16}$.

Solution. We know that the polar form of $-1 - i$ is $\sqrt{2}e^{-3\pi/4}$ and so

$$(-1 - i)^{16} = \left(\sqrt{2}\right)^{16} e^{-16i(3\pi/4)} = 2^8 e^{-12i\pi} = 256(e^{i\pi})^{-12} = 256(-1)^{-12} = 256$$

using the previous theorem along with Euler's formula. Thus, $\mathrm{Re}\left((-1 - i)^{16}\right) = 256$ and $\mathrm{Im}\left((-1 - i)^{16}\right) = 0$.