Lecture #4: Polar Form of a Complex Variable

Suppose that $z = x + iy$ is a complex variable. Our goal is to define the polar form of a complex variable. We start by describing how our experience with real variables motivates this definition.

Consider the pair $(x, y) \in \mathbb{R}^2$ in cartesian coordinates. We know from Math 213 that an equivalent way to describe a point in the plane is in terms of polar coordinates. That is, we can describe the point (x, y) in terms of its distance r from the origin and the angle the point makes with the positive x-axis. This leads to the change-of-variables

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$

where $r \geq 0$ and $0 \leq \theta < 2\pi$. If we try to invert this transformation and solve for r and θ, then we find

$$r = \sqrt{x^2 + y^2} \quad \text{and} \quad \theta = \arctan(y/x).$$

The trouble here is that the inverse equation

$$\theta = \arctan(y/x)$$

is not true for pairs (x, y) in the second or third quadrants. The reason for this is the *convention* that the standard interpretation of the arctangent function places its range in the first and fourth quadrants; that is, by convention, the domain of the tangent function is restricted to $(-\pi/2, \pi/2)$ in order for the inverse of tangent function to be single-valued. Note the reason for this convention. The only asymptotes of the tangent function on $(-\pi/2, \pi/2)$ are at the endpoints. If instead we considered the tangent function on the interval $[0, \pi]$, then we would have the issue that the tangent function is not defined at $\pi/2$. This would then lead to the domain of the tangent function being $[0, \pi/2) \cup (\pi/2, \pi]$ which is ugly. The conclusion is that we cannot define θ simply as $\theta = \arctan(y/x)$.

Thus, when we are working with real variables (in particular in Math 213), we define θ as the *unique* angle $\theta \in [0, 2\pi)$ satisfying

$$\cos\theta = \frac{x}{\sqrt{x^2 + y^2}} \quad \text{and} \quad \sin\theta = \frac{y}{\sqrt{x^2 + y^2}}.$$

For complex variables, however, we follow a different convention. While it is true that there is a unique angle θ in *any* half-open half-closed interval of length 2π satisfying

$$\cos\theta = \frac{x}{\sqrt{x^2 + y^2}} \quad \text{and} \quad \sin\theta = \frac{y}{\sqrt{x^2 + y^2}},$$

we must make a *convention* as to which choice of definite interval we wish to make. We declare $(-\pi, \pi]$ as our convention for complex variables.

Definition. Suppose that $z = x + iy \in \mathbb{C}$, $z \neq 0$. Define the *argument* of z, denoted $\arg z$, to be *any* solution θ of the pair of equations

$$\cos\theta = \frac{x}{|z|} \quad \text{and} \quad \sin\theta = \frac{y}{|z|}.$$

Note that if θ_0 qualifies as a value of $\arg z$, then so do

$$\theta_0 \pm 2\pi, \theta_0 \pm 4\pi, \theta_0 \pm 6\pi, \ldots.$$

Moreover, every value of $\arg z$ must be one of these.

Remark. If $z = 0$, then we take, by convention, $\arg 0 = \{0, \pm 2\pi, \pm 4\pi, \ldots\}$.

However, we still have the problem of multi-valuedness. For definiteness, we will want only a single value of the argument. This leads to the following definition.

Definition. Suppose that $z = x + iy \in \mathbb{C}$. Define the *principal value of the argument* of z, denoted $\operatorname{Arg} z$, to be the unique value of $\arg z \in (-\pi, \pi]$.

In particular, $\operatorname{Arg} 0 = 0$.

There is a more sophisticated reason for the convention that $\operatorname{Arg} z \in (-\pi, \pi]$ than just trying to avoid $[0, 2\pi)$. This has to do with the definition of square root. We will want to maintain the convention that the square root of a positive real number is a positive real number. This is easiest to achieve if we choose $\operatorname{Arg} z \in (-\pi, \pi]$. We will be discussing this point in much detail later in the course.

Definition. Suppose that $z \in \mathbb{C}$. We define the *polar form* of z to be $re^{i\theta}$ where $r = |z|$ and $\theta = \operatorname{Arg} z$. For convenience, we will write $z = re^{i\theta}$.

Example 4.1. Write $z = 1 + i$ in polar form and identify $\arg z$.

Solution. If $z = 1 + i$, then

$$|z| = \sqrt{1^2 + 1^2} = \sqrt{2} = r.$$

Moreover,

$$\cos\theta = \frac{1}{\sqrt{2}} \quad \text{and} \quad \sin\theta = \frac{1}{\sqrt{2}}$$

implies that

$$\theta = \frac{\pi}{4} + 2\pi k$$

for $k \in \mathbb{Z}$. Thus,

$$\operatorname{Arg} z = \frac{\pi}{4} \quad \text{and} \quad \arg z = \left\{\frac{\pi}{4}, \frac{\pi}{4} \pm 2\pi, \frac{\pi}{4} \pm 4\pi, \ldots\right\} = \left\{\frac{\pi}{4} + 2\pi k : k \in \mathbb{Z}\right\}.$$

Hence, the polar form of $z = 1 + i$ is

$$\sqrt{2}e^{i\pi/4}.$$