Lecture #33: Computing Real Improper Integrals

Example 33.1. Compute

$$\int_0^\infty \frac{x^2}{(x^2+1)(x^2+4)}\,\mathrm{d}x.$$

Solution. Although it is possible to compute this particular integral using partial fractions easily enough, we will solve it with complex variables in order to illustrate a general method which works in more complicated cases. Observe that by symmetry,

$$2\int_0^\infty \frac{x^2}{(x^2+1)(x^2+4)}\,\mathrm{d}x = \int_{-\infty}^\infty \frac{x^2}{(x^2+1)(x^2+4)}\,\mathrm{d}x.$$

Suppose that $C = C_R \oplus [-R,R]$ denotes the closed contour oriented counterclockwise obtained by concatenating C_R, that part of the circle of radius R in the upper half plane parametrized by $z(t) = Re^{it}$, $0 \le t \le \pi$, with $[-R,R]$, the line segment along the real axis connecting the point $-R$ to the point R. Therefore, if

$$f(z) = \frac{z^2}{(z^2+1)(z^2+4)},$$

then

$$\int_C f(z)\,\mathrm{d}z = \int_{[-R,R]} f(z)\,\mathrm{d}z + \int_{C_R} f(z)\,\mathrm{d}z. \tag{$*$}$$

We now observe that we can compute

$$\int_C f(z)\,\mathrm{d}z = \int_C \frac{z^2}{(z^2+1)(z^2+4)}\,\mathrm{d}z$$

using the Residue Theorem. That is, since

$$f(z) = \frac{z^2}{(z^2+1)(z^2+4)} = \frac{z^2}{(z+i)(z-i)(z+2i)(z-2i)},$$

we find that $f(z)$ has simple poles at $z_1 = i$, $z_2 = -i$, $z_3 = 2i$, $z_4 = -2i$. However, only z_1 and z_3 are inside C (assuming, of course, that R is sufficiently large). Thus,

$$\operatorname{Res}(f;z_1) = \left.\frac{z^2}{(z+i)(z+2i)(z-2i)}\right|_{z=z_1=i} = \frac{i^2}{(2i)(3i)(-i)} = \frac{i}{6}$$

and

$$\operatorname{Res}(f;z_3) = \left.\frac{z^2}{(z+i)(z-i)(z+2i)}\right|_{z=z_3=2i} = \frac{(2i)^2}{(3i)(i)(4i)} = -\frac{i}{3},$$

so by the Residue Theorem,

$$\int_C f(z)\,\mathrm{d}z = \int_C \frac{z^2}{(z^2+1)(z^2+4)}\,\mathrm{d}z = 2\pi i\left(\frac{i}{6} - \frac{i}{3}\right) = \frac{\pi}{3}.$$

In other words, we have shown that for R sufficiently large $(*)$ becomes

$$\frac{\pi}{3} = \int_{[-R,R]} \frac{z^2}{(z^2+1)(z^2+4)}\,\mathrm{d}z + \int_{C_R} \frac{z^2}{(z^2+1)(z^2+4)}\,\mathrm{d}z.$$

The next step is to observe that since $[-R,R]$ denotes the line segment along the real axis connecting the point $-R$ to the point R, if we parametrize the line segment by $z(t)=t$, $-R \le t \le R$, then since $z'(t)=1$, we obtain

$$\int_{[-R,R]} \frac{z^2}{(z^2+1)(z^2+4)}\,\mathrm{d}z = \int_{-R}^{R} \frac{t^2}{(t^2+1)(t^2+4)}\,\mathrm{d}t = \int_{-R}^{R} \frac{x^2}{(x^2+1)(x^2+4)}\,\mathrm{d}x$$

where the last equality follows by a simple change of dummy variable from t to x. Thus,

$$\frac{\pi}{3} = \int_{-R}^{R} \frac{x^2}{(x^2+1)(x^2+4)}\,\mathrm{d}x + \int_{C_R} \frac{z^2}{(z^2+1)(z^2+4)}\,\mathrm{d}z$$

and so by taking the limit as $R \to \infty$ of both sides we obtain

$$\frac{\pi}{3} = \lim_{R\to\infty} \int_{-R}^{R} \frac{x^2}{(x^2+1)(x^2+4)}\,\mathrm{d}x + \lim_{R\to\infty} \int_{C_R} \frac{z^2}{(z^2+1)(z^2+4)}\,\mathrm{d}z.$$

We now make two claims.

Claim 1.

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{x^2}{(x^2+1)(x^2+4)}\,\mathrm{d}x = \int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}\,\mathrm{d}x$$

Claim 2.

$$\lim_{R\to\infty} \int_{C_R} \frac{z^2}{(z^2+1)(z^2+4)}\,\mathrm{d}z = 0$$

Assuming that both claims are true, we obtain

$$\frac{\pi}{3} = \int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}\,\mathrm{d}x \quad \text{and so} \quad \int_{0}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}\,\mathrm{d}x = \frac{\pi}{6}$$

which is in agreement with what one obtains by using partial fractions.

Hence, the next task is to address these two claims. We will begin with the second claim which follows immediately from this result.

Theorem 33.2. *Suppose that C_R denotes the upper half of the circle connecting R to $-R$ and parametrized by $z(t)=Re^{it}$, $0 \le t \le \pi$. If*

$$f(z) = \frac{P(z)}{Q(z)}$$

is the ratio of two polynomials satisfying $\deg Q \ge \deg P + 2$, then

$$\lim_{R\to\infty} \int_{C_R} f(z) = 0.$$

Proof. The fact that $\deg Q \geq \deg P + 2$ implies that if $|z|$ is sufficiently large, then

$$|f(z)| = \left|\frac{P(z)}{Q(z)}\right| \leq \frac{K}{|z|^2} \tag{$\dagger$}$$

for some constant $K < \infty$. (See the supplement for derivation of (†).) Hence,

$$\left|\int_{C_R} f(z)\,\mathrm{d}z\right| \leq \int_{C_R} |f(z)|\,|\mathrm{d}z| = \int_{C_R} \frac{K}{|z|^2}\,|\mathrm{d}z| = \frac{K}{R^2}\ell(C_R) = \frac{K\pi}{R}$$

since $\ell(C_R) = \pi R$ is the arclength of C_R. Taking $R \to \infty$ then yields the result. $\square$

Thus, we conclude from this theorem that

$$\lim_{R\to\infty} \int_{C_R} \frac{z^2}{(z^2+1)(z^2+4)}\,\mathrm{d}z = 0$$

since $P(z) = z^2$ has degree 2 and $Q(z) = (z^2+1)(z^2+4)$ has degree 4.

Review of Improper Integrals

In order to discuss Claim 1, it is necessary to review improper integrals from first year calculus. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous. By definition,

$$\int_0^\infty f(x)\,\mathrm{d}x = \lim_{b\to\infty} \int_0^b f(x)\,\mathrm{d}x.$$

Thus, assuming the limit exists as a real number, we define

$$\int_0^\infty f(x)\,\mathrm{d}x$$

to be this limiting value. By definition,

$$\int_{-\infty}^0 f(x)\,\mathrm{d}x = \lim_{a\to-\infty} \int_a^0 f(x)\,\mathrm{d}x.$$

Thus, assuming the limit exists as a real number, we define

$$\int_{-\infty}^0 f(x)\,\mathrm{d}x$$

to be this limiting value. By definition,

$$\int_{-\infty}^\infty f(x)\,\mathrm{d}x = \int_0^\infty f(x)\,\mathrm{d}x + \int_{-\infty}^0 f(x)\,\mathrm{d}x = \lim_{b\to\infty} \int_0^b f(x)\,\mathrm{d}x + \lim_{a\to-\infty} \int_a^0 f(x)\,\mathrm{d}x.$$

Thus, in order for

$$\int_{-\infty}^\infty f(x)\,\mathrm{d}x \tag{$*$}$$

to exist it must be the case that both

$$\lim_{b\to\infty}\int_0^b f(x)\,\mathrm{d}x \quad\text{and}\quad \lim_{a\to-\infty}\int_a^0 f(x)\,\mathrm{d}x$$

exist as real numbers. However, if one of these limits fails to exist as a real number, then the improper integral $(*)$ does not exist. Sometimes, we might write

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \lim_{a\to-\infty, b\to\infty}\int_a^b f(x)\,\mathrm{d}x \tag{$\dagger$}$$

instead which just writes the two separate limits in a single piece of notation. It is important to stress that this notation still implies that two separate limits are being taken: $a \to -\infty$ and $b \to \infty$. It might be tempting to try and combine the two separate limits into a single limit as follows:

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \lim_{c\to\infty}\int_{-c}^{c} f(x)\,\mathrm{d}x. \tag{$\ddagger$}$$

However, $(\dagger)$ and $(\ddagger)$ are not the same! As we will now show, it is possible for

$$\lim_{c\to\infty}\int_{-c}^{c} f(x)\,\mathrm{d}x$$

to exist, but for

$$\lim_{a\to-\infty, b\to\infty}\int_a^b f(x)\,\mathrm{d}x$$

not to exist.

Example 33.3. Observe that

$$\int_{-c}^{c}\frac{x}{1+x^2}\,\mathrm{d}x = \left.\frac{\log(1+x^2)}{2}\right|_{-c}^{c} = \frac{\log(1+c^2)}{2} - \frac{\log(1+c^2)}{2} = 0$$

and so

$$\lim_{c\to\infty}\int_{-c}^{c}\frac{x}{1+x^2}\,\mathrm{d}x = \lim_{c\to 0} 0 = 0.$$

On the other hand,

$$\int_a^0 \frac{x}{1+x^2}\,\mathrm{d}x = \left.\frac{\log(1+x^2)}{2}\right|_a^0 = -\frac{\log(1+a^2)}{2} \quad\text{and}\quad \int_0^b \frac{x}{1+x^2}\,\mathrm{d}x = \left.\frac{\log(1+x^2)}{2}\right|_0^b = \frac{\log(1+b^2)}{2}$$

so that

$$\lim_{a\to-\infty}\int_a^0 \frac{x}{1+x^2}\,\mathrm{d}x = -\lim_{a\to-\infty}\frac{\log(1+a^2)}{2} = -\infty \quad\text{and}\quad \lim_{b\to\infty}\int_0^b \frac{x}{1+x^2}\,\mathrm{d}x = \lim_{b\to\infty}\frac{\log(1+b^2)}{2} = \infty.$$

Thus,

$$\lim_{a\to-\infty, b\to\infty}\int_a^b \frac{x}{1+x^2}\,\mathrm{d}x = -\infty + \infty = \infty - \infty$$

so that

$$\int_{-\infty}^{\infty}\frac{x}{1+x^2}\,\mathrm{d}x$$

does not exist.

Supplement: Verification of (†) from proof of Theorem 33.2

Suppose that $Q(z) = b_0 + b_1 z + \cdots + b_m z^m$ is a polynomial of degree m. Without loss of generality assume that $b_m = 1$. Therefore,

$$z^{-m}Q(z) = 1 + \frac{b_{m-1}}{z} + \cdots + \frac{b_0}{z^m}$$

and so by the triangle inequality,

$$|z^{-m}||Q(z)| = \left|1 + \frac{b_{m-1}}{z} + \cdots + \frac{b_0}{z^m}\right| \geq 1 - \left|\frac{b_{m-1}}{z} + \cdots + \frac{b_0}{z^m}\right|.$$

Let $M = \max\{1, |b_0|, \ldots, |b_{m-1}|\}$ and note that $2mM > 1$. This means that if $|z| \geq 2mM$, then

$$\left|\frac{b_{m-j}}{z^j}\right| \leq \frac{M}{|z|^j} \leq \frac{M}{|z|} \leq \frac{1}{2m}.$$

Therefore, since there are m terms in the following sum,

$$\left|\frac{b_{m-1}}{z} + \cdots + \frac{b_0}{z^m}\right| \leq \frac{1}{2m} + \frac{1}{2m} + \cdots + \frac{1}{2m} = \frac{1}{2}$$

which implies that

$$|z^{-m}||Q(z)| \geq 1 - \left|\frac{b_{m-1}}{z} + \cdots + \frac{b_0}{z^m}\right| \geq 1 - \frac{1}{2} = \frac{1}{2}.$$

Hence, we obtain,

$$|Q(z)| \geq \frac{|z|^m}{2}$$

for $|z|$ sufficiently large. On the other hand, suppose that $P(z) = a_0 + a_1 z + \cdots + a_n z^n$ is a polynomial of degree n so that

$$z^{-n}P(z) = a_n + \frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n}.$$

If $|z| > 1$, then by the triangle inequality,

$$|z^{-n}P(z)| \leq |a_n| + \left|\frac{a_{n-1}}{z}\right| + \cdots + \left|\frac{a_0}{z^n}\right| \leq |a_n| + |a_{n-1}| + \cdots + |a_0|$$

and so with $C = |a_0| + \cdots + |a_n|$ we obtain

$$|z^{-n}P(z)| \leq C$$

for $|z| > 1$. Now suppose that

$$f(z) = \frac{P(z)}{Q(z)}$$

is the ratio of polynomials with $\deg Q(z) \geq \deg P(z) + 2$. If $\deg P(z) = n$ and $\deg Q(z) = n + k$ with $k \geq 2$, then we find that for $|z|$ sufficiently large,

$$\left|\frac{P(z)}{Q(z)}\right| = \frac{|z^{-n}P(z)|}{|z^{-n}Q(z)|} \leq \frac{C}{|z|^{-n}\frac{|z|^{n+k}}{2}} = \frac{2C}{|z|^k} \leq \frac{2C}{|z|^2} = \frac{K}{|z|^2}$$

since $k \geq 2$.