Lecture #32: Computing Real Trigonometric Integrals

Suppose that C is a closed contour oriented counterclockwise. Last class we proved the Residue Theorem which states that if $f(z)$ is analytic inside and on C, except for a finite number of isolated singular points $z_1, \ldots, z_n$, then

$$\int_C f(z)\,\mathrm{d}z = 2\pi i \sum_{j=1}^{n} \operatorname{Res}(f; z_j).$$

In order to compute $\operatorname{Res}(f; z_j)$, we need to determine whether or not the isolated singular point z_j is removable, essential, or a pole of order m. If z_j is a pole of order m, then we know

$$\operatorname{Res}(f; z_j) = \frac{1}{(m-1)!} \frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}} (z - z_j)^m f(z)\bigg|_{z=z_j}.$$

In particular, if z_j is a simple pole, then

$$\operatorname{Res}(f; z_j) = (z - z_j) f(z)\bigg|_{z=z_j}.$$

Example 32.1. Compute

$$\int_C \frac{2}{z^2 + 4z + 1}\,\mathrm{d}z$$

where $C = \{|z| = 1\}$ is the circle of radius 1 centred at 0 oriented counterclockwise.

Solution. Clearly

$$f(z) = \frac{2}{z^2 + 4z + 1}$$

has two simple poles. Notice that $z^2 + 4z + 1 = (z^2 + 4z + 4) - 3 = (z + 2)^2 - 3 = 0$ implies

$$z_1 = \sqrt{3} - 2 \quad \text{and} \quad z_2 = -\sqrt{3} - 2$$

are simple poles. However, $|z_1| < 1$ and $|z_2| > 1$ which means that $f(z)$ only has one isolated singularity inside C. Since

$$z^2 + 4z + 1 = (z - z_1)(z - z_2) = (z - \sqrt{3} + 2)(z + \sqrt{3} + 2),$$

we find

$$\operatorname{Res}(f; z_1) = \operatorname{Res}(f; \sqrt{3} - 2) = \frac{2}{z + \sqrt{3} + 2}\bigg|_{z=\sqrt{3}-2} = \frac{2}{2\sqrt{3}} = \frac{1}{\sqrt{3}},$$

so by the Cauchy Residue Theorem,

$$\int_C \frac{2z}{z^2 + 4z + 1}\,\mathrm{d}z = 2\pi i \cdot \frac{1}{\sqrt{3}} = \frac{2}{\sqrt{3}}\pi i.$$

Suppose that we now parametrize C by $z(t) = e^{it}$, $0 \le t \le 2\pi$, and attempt to compute this same contour integral using

$$\int_C f(z)\,\mathrm{d}z = \int_0^{2\pi} f(e^{it}) \cdot ie^{it}\,\mathrm{d}t.$$

That is,

$$\begin{aligned}\int_C \frac{2}{z^2+4z+1}\,\mathrm{d}z = \int_0^{2\pi} \frac{2}{e^{2it}+4e^{it}+1} \cdot ie^{it}\,\mathrm{d}t &= 2i \int_0^{2\pi} \frac{e^{it}}{e^{2it}+4e^{it}+1}\,\mathrm{d}t \\ &= 2i \int_0^{2\pi} \frac{1}{e^{it}+4+e^{-it}}\,\mathrm{d}t \\ &= 2i \int_0^{2\pi} \frac{1}{4+2\cos t}\,\mathrm{d}t \\ &= i \int_0^{2\pi} \frac{1}{2+\cos t}\,\mathrm{d}t\end{aligned}$$

and so

$$\frac{2}{\sqrt{3}}\pi i = i\int_0^{2\pi} \frac{1}{2+\cos t}\,\mathrm{d}t \quad \text{or, equivalently,} \quad \int_0^{2\pi} \frac{1}{2+\cos t}\,\mathrm{d}t = \frac{2}{\sqrt{3}}\pi.$$

Notice that we were able to compute a definite integral by relating it to a contour integral that could be evaluated using the Residue Theorem. If we have a definite integral, the limits of integration are 0 to 2π, and the integrand is a function of $\cos\theta$ and $\sin\theta$, then we can systematically convert it to a contour integral as follows. Let $C = \{|z| = 1\}$ denote the circle of radius 1 centred at 0 oriented counterclockwise and parametrize C by $z(\theta) = e^{i\theta}$, $0 \le \theta \le 2\pi$, so that $z'(\theta) = ie^{i\theta} = iz(\theta)$. Since

$$z(\theta) = e^{i\theta} = \cos\theta + i\sin\theta \quad \text{and} \quad \frac{1}{z(\theta)} = e^{-i\theta} = \cos\theta - i\sin\theta,$$

we find

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{1}{2}\left(z(\theta) + \frac{1}{z(\theta)}\right) \quad \text{and} \quad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{1}{2i}\left(z(\theta) - \frac{1}{z(\theta)}\right).$$

Therefore,

$$\begin{aligned}\int_0^{2\pi} F(\cos\theta, \sin\theta)\,\mathrm{d}\theta &= \int_0^{2\pi} F\left(\frac{1}{2}\left(z(\theta)+\frac{1}{z(\theta)}\right), \frac{1}{2i}\left(z(\theta)-\frac{1}{z(\theta)}\right)\right)\mathrm{d}\theta \\ &= \int_0^{2\pi} F\left(\frac{1}{2}\left(z(\theta)+\frac{1}{z(\theta)}\right), \frac{1}{2i}\left(z(\theta)-\frac{1}{z(\theta)}\right)\right)\frac{z'(\theta)}{z'(\theta)}\,\mathrm{d}\theta \\ &= \int_0^{2\pi} F\left(\frac{1}{2}\left(z(\theta)+\frac{1}{z(\theta)}\right), \frac{1}{2i}\left(z(\theta)-\frac{1}{z(\theta)}\right)\right)\frac{1}{iz(\theta)}z'(\theta)\,\mathrm{d}\theta \\ &= \int_C F\left(\frac{1}{2}\left(z+\frac{1}{z}\right), \frac{1}{2i}\left(z-\frac{1}{z}\right)\right)\frac{1}{iz}\,\mathrm{d}z.\end{aligned}$$

Example 32.2. Compute

$$\int_0^{2\pi} \frac{\cos(2\theta)}{5 - 4\cos\theta}\,\mathrm{d}\theta.$$

Solution. Let $C = \{|z| = 1\}$ oriented counterclockwise be parametrized by $z(\theta) = e^{i\theta}$, $0 \le \theta \le 2\pi$. Note that

$$\cos(2\theta) = \frac{e^{2i\theta} + e^{-2i\theta}}{2} = \frac{1}{2}\left(z(\theta)^2 + \frac{1}{z(\theta)^2}\right) = \frac{z(\theta)^4 + 1}{2z(\theta)^2}$$

and

$$5 - 4\cos\theta = 5 - 4 \cdot \frac{1}{2}\left(z(\theta) + \frac{1}{z(\theta)}\right) = 5 - 2z(\theta) - \frac{2}{z(\theta)} = -\frac{2z(\theta)^2 - 5z(\theta) + 2}{z(\theta)},$$

so that

$$\frac{\cos(2\theta)}{5 - 4\cos\theta} = \frac{\frac{z(\theta)^4+1}{2z(\theta)^2}}{-\frac{2z(\theta)^2 - 5z(\theta)+2}{z(\theta)}} = -\frac{z(\theta)^4 + 1}{2z(\theta)\left(2z(\theta)^2 - 5z(\theta) + 2\right)} = -\frac{z(\theta)^4 + 1}{2z(\theta)(2z(\theta) - 1)(z(\theta) - 2)}.$$

This implies

$$\begin{aligned}\int_0^{2\pi} \frac{\cos(2\theta)}{5 - 4\cos\theta}\,\mathrm{d}\theta = -\int_0^{2\pi} \frac{z(\theta)^4 + 1}{2z(\theta)(2z(\theta) - 1)(z(\theta) - 2)}\,\mathrm{d}\theta &= -\int_C \frac{z^4 + 1}{2z(2z - 1)(z - 2)} \cdot \frac{1}{iz}\,\mathrm{d}z \\ &= \frac{i}{2}\int_C \frac{z^4 + 1}{z^2(2z - 1)(z - 2)}\,\mathrm{d}z.\end{aligned}$$

Let

$$f(z) = \frac{z^4 + 1}{z^2(2z - 1)(z - 2)}$$

so that $f(z)$ clearly has a double pole at $z_1 = 0$, a simple pole at $z_2 = 1/2$, and a simple pole at $z_3 = 2$. Of these three singularities of $f(z)$, only two of them are inside C. Therefore,

$$\mathrm{Res}(f; 0) = \frac{\mathrm{d}}{\mathrm{d}z}\frac{z^4 + 1}{(2z - 1)(z - 2)}\bigg|_{z=0} = \frac{4z^3(2z - 1)(z - 2) - (z^4 + 1)(4z - 5)}{(2z - 1)^2(z - 2)^2}\bigg|_{z=0} = \frac{5}{4}$$

and

$$\mathrm{Res}(f; 1/2) = \left(z - \frac{1}{2}\right)\frac{z^4 + 1}{z^2(2z - 1)(z - 2)}\bigg|_{z=1/2} = \frac{z^4 + 1}{2z^2(z - 2)}\bigg|_{z=1/2} = -\frac{17}{12}.$$

By the Residue Theorem,

$$\int_0^{2\pi} \frac{\cos(2\theta)}{5 - 4\cos\theta}\,\mathrm{d}\theta = \frac{i}{2}\int_C \frac{z^4 + 1}{z^2(2z - 1)(z - 2)}\,\mathrm{d}z = \frac{i}{2} \cdot 2\pi i\left(\frac{5}{4} - \frac{17}{12}\right) = \frac{\pi}{6}.$$

Remark. It is possible to compute both

$$\int_0^{2\pi} \frac{1}{2 + \cos\theta}\,\mathrm{d}\theta \quad \text{and} \quad \int_0^{2\pi} \frac{\cos(2\theta)}{5 - 4\cos\theta}\,\mathrm{d}\theta$$

by calculating indefinite Riemann integrals. However, such calculations are *very* challenging. The contour integral approach is significantly easier.