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Lecture #31: The Cauchy Residue Theorem

Recall that last class we showed that a function $f(z)$ has a pole of order m at z_0 if and only if

$$f(z) = \frac{g(z)}{(z - z_0)^m}$$

for some function $g(z)$ that is analytic in a neighbourhood of z_0 and has $g(z_0) \neq 0$. We also derived a formula for $\mathrm{Res}(f; z_0)$.

Theorem 31.1. *If $f(z)$ is analytic for $0 < |z - z_0| < R$ and has a pole of order m at z_0, then*

$$\mathrm{Res}(f; z_0) = \frac{1}{(m-1)!} \frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}} (z - z_0)^m f(z) \bigg|_{z=z_0} = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}} (z - z_0)^m f(z).$$

In particular, if z_0 is a simple pole, then

$$\mathrm{Res}(f; z_0) = (z - z_0) f(z) \bigg|_{z=z_0} = \lim_{z \to z_0} (z - z_0) f(z).$$

Example 31.2. Suppose that

$$f(z) = \frac{\sin z}{(z^2 - 1)^2}.$$

Determine the order of the pole at $z_0 = 1$.

Solution. Observe that $z^2 - 1 = (z-1)(z+1)$ and so

$$f(z) = \frac{\sin z}{(z^2 - 1)^2} = \frac{\sin z}{(z-1)^2 (z+1)^2} = \frac{\sin z/(z+1)^2}{(z-1)^2}.$$

Since

$$g(z) = \frac{\sin z}{(z+1)^2}$$

is analytic at 1 and $g(1) = 2^{-2} \sin(1) \neq 0$, we conclude that $z_0 = 1$ is a pole of order 2.

Example 31.3. Determine the residue at $z_0 = 1$ of

$$f(z) = \frac{\sin z}{(z^2 - 1)^2}$$

and compute

$$\int_C f(z) \, \mathrm{d}z$$

where $C = \{|z - 1| = 1/2\}$ is the circle of radius 1/2 centred at 1 oriented counterclockwise.

Solution. Since we can write $(z-1)^2 f(z) = g(z)$ where

$$g(z) = \frac{\sin z}{(z+1)^2}$$

is analytic at $z_0 = 1$ with $g(1) \neq 0$, the residue of $f(z)$ at $z_0 = 1$ is

$$\begin{aligned}
\operatorname{Res}(f;1) = \frac{1}{(2-1)!}\frac{\mathrm{d}^{2-1}}{\mathrm{d}z^{2-1}}(z-1)^2 f(z)\bigg|_{z=1} &= \frac{\mathrm{d}}{\mathrm{d}z}(z-1)^2 f(z)\bigg|_{z=1} \\
&= \frac{\mathrm{d}}{\mathrm{d}z}\frac{\sin z}{(z+1)^2}\bigg|_{z=1} \\
&= \frac{(z+1)^2\cos z - 2(z+1)\sin z}{(z+1)^4}\bigg|_{z=1} \\
&= \frac{4\cos 1 - 4\sin 1}{16} \\
&= \frac{\cos 1 - \sin 1}{4}.
\end{aligned}$$

Observe that if $C = \{|z-1| = 1/2\}$ oriented counterclockwise, then the only singularity of $f(z)$ inside C is at $z_0 = 1$. Therefore,

$$\int_C \frac{\sin z}{(z^2-1)^2}\,\mathrm{d}z = 2\pi i \operatorname{Res}(f;1) = \frac{(\cos 1 - \sin 1)\pi i}{2}.$$

It is worth pointing out that we could have also obtained this solution using the Cauchy Integral Formula; that is,

$$\int_C \frac{\sin z}{(z^2-1)^2}\,\mathrm{d}z = \int_C \frac{g(z)}{(z-1)^2}\,\mathrm{d}z = 2\pi i g'(1) = 2\pi i \cdot \frac{\cos 1 - \sin 1}{4} = \frac{(\cos 1 - \sin 1)\pi i}{2}$$

as above.

Remark. Suppose that C is a closed contour oriented counterclockwise. If $f(z)$ is analytic inside and on C except for a single point z_0 where $f(z)$ has a pole of order m, then both the Cauchy Integral Formula and the residue formula will require exactly the same work, namely the calculation of the $m-1$ derivative of $(z-z_0)^m f(z)$.

Recall that there are two other types of isolated singular points to consider, namely removable singularities and essential singularities. If the singularity is removable, then the residue is obviously 0. Unfortunately, there is no direct way to determine the residue associated with an essential singularity. The coefficient a_{-1} of the Laurent series must be determined explicitly.

In summary, suppose that $f(z)$ is analytic for $0 < |z-z_0| < R$ and has an isolated singularity at z_0. By direct inspection of the function, one may make an educated guess as to whether the isolated singularity is removable, a pole, or essential. If it believed to be either removable or essential, then compute the Laurent series to determine $\operatorname{Res}(f;z_0)$. If it is believed to be a pole, then attempt to compute $\operatorname{Res}(f;z_0)$ using Theorem 30.6.

Theorem 31.4 (Cauchy Residue Theorem)**.** *Suppose that C is a closed contour oriented counterclockwise. If $f(z)$ is analytic inside and on C except at a finite number of isolated singularities $z_1, z_2, \ldots, z_n$, then*

$$\int_C f(z)\,\mathrm{d}z = 2\pi i \sum_{j=1}^{n} \operatorname{Res}(f; z_j).$$

Proof. Observe that if C is a closed contour oriented counterclockwise, then integration over C can be continuously deformed to a union of integrations over $C_1, C_2, \ldots, C_n$ where C_j is a circle oriented counterclockwise encircling exactly one isolated singularity, namely z_j, and not passing through any of the other isolated singular points. This yields

$$\int_C f(z)\,\mathrm{d}z = \int_{C_1} f(z)\,\mathrm{d}z + \cdots + \int_{C_n} f(z)\,\mathrm{d}z.$$

Since

$$\int_{C_j} f(z)\,\mathrm{d}z = 2\pi i \operatorname{Res}(f; z_j),$$

the proof is complete. □

Remark. Note that if the isolated singularities of $f(z)$ inside C are all either removable or poles, then the Cauchy Integral Formula can also be used to compute

$$\int_C f(z)\,\mathrm{d}z.$$

If any of the isolated singularities are essential, then the Cauchy Integral Formula does not apply. Moreover, even when $f(z)$ has only removable singularities or poles, the Residue Theorem is often much easier to use than the Cauchy Integral Formula.

Example 31.5. Compute

$$\int_C \frac{3z^3 + 4z^2 - 5z + 1}{(z - 2i)(z^3 - z)}\,\mathrm{d}z$$

where $C = \{|z| = 3\}$ is the circle of radius 3 centred at 0 oriented counterclockwise.

Solution. Observe that

$$f(z) = \frac{3z^3 + 4z^2 - 5z + 1}{(z - 2i)(z^3 - z)} = \frac{3z^3 + 4z^2 - 5z + 1}{z(z-1)(z+1)(z-2i)}$$

has isolated singular points at $z_1 = 0$, $z_2 = 1$, $z_3 = -1$, and $z_4 = 2i$. Moreover, each isolated singularity is a simple pole, and so

$$\operatorname{Res}(f; 0) = \left.\frac{3z^3 + 4z^2 - 5z + 1}{(z-1)(z+1)(z-2i)}\right|_{z=0} = \frac{1}{-1 \cdot -2i} = -\frac{i}{2},$$

$$\operatorname{Res}(f; 1) = \left.\frac{3z^3 + 4z^2 - 5z + 1}{z(z+1)(z-2i)}\right|_{z=1} = \frac{3 + 4 - 5 + 1}{1 \cdot 2 \cdot (1 - 2i)} = \frac{3}{2(1-2i)} = \frac{3(1+2i)}{10},$$

$$\operatorname{Res}(f;-1) = \left.\frac{3z^3+4z^2-5z+1}{z(z-1)(z-2i)}\right|_{z=-1} = \frac{-3+4+5+1}{-1\cdot -2\cdot(-1-2i)} = -\frac{7}{2(1+2i)} = \frac{7(2i-1)}{10},$$

$$\operatorname{Res}(f;2i) = \left.\frac{3z^3+4z^2-5z+1}{z(z-1)(z+1)}\right|_{z=2i} = \frac{3(2i)^3+4(2i)^2-5(2i)+1}{2i(2i-1)(2i+1)} = \frac{34-15i}{10}.$$

By the Cauchy Residue Theorem,

$$\int_C \frac{3z^3+4z^2-5z+1}{(z-2i)(z^3-z)}\,\mathrm{d}z = 2\pi i\left(-\frac{i}{2}+\frac{3(1+2i)}{10}+\frac{7(2i-1)}{10}+\frac{34-15i}{10}\right) = 6\pi i.$$