Mathematics 312 (Fall 2013) September 6, 2013
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Lecture #2: Algebraic Properties of $\mathbb{C}$

Recall that $z = a + ib$, with $i = \sqrt{-1}$ and $a, b \in \mathbb{R}$, is a *complex variable.*

Cartesian Representation (or Geometric Interpretation) of Complex Variables

We can represent the complex variable $z = a + ib$ as the point in the plane (a, b) as shown in Figure 2.1.

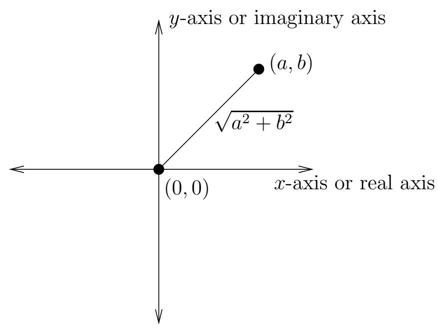

Figure 2.1: The identification of $\mathbb{C}$ with $\mathbb{R}^2$.

Note. In other words, if we let $\mathbb{C} = \{z = a+ib : a, b \in \mathbb{R}\}$ denote the set of complex variables, then we can identify $\mathbb{C}$ with the two-dimensional cartesian plane $\mathbb{R}^2$ via the identification

$$z = a + ib \in \mathbb{C} \longleftrightarrow (a, b) \in \mathbb{R}^2.$$

This identification is actually an *isomorphism* and so an algebraist might say that $\mathbb{C}$ and $\mathbb{R}^2$ are *isomorphic* and write $\mathbb{C} \cong \mathbb{R}^2$. We will not be concerned with isomorphisms in this class.

Observe that the distance from the point (a, b) in the plane to the origin $(0, 0)$ is

$$\sqrt{a^2 + b^2}$$

as shown in Figure 2.1. This motivates the following definition.

Definition. Let $z = a + ib$ be a complex variable. The *modulus* or *absolute value* of z, denoted $|z|$, is defined as

$$|z| = \sqrt{a^2 + b^2}.$$

Definition. Let $z = a + ib$ be a complex variable. The *(complex) conjugate* of z, denoted $\bar{z}$, is defined as

$$\bar{z} = a - ib.$$

Exercise 2.1. Suppose that z is a complex variable. Show that $z\bar{z} = |z|^2$.

Geometrically, conjugation represents reflection in the real axis; see Figure 2.2.

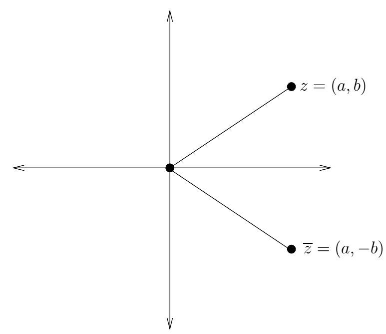

Figure 2.2: Geometric representation of complex conjugation.

Proposition 2.2. *If $z = a + ib$ is a complex variable, then $\sqrt{z\,\overline{z}}$ is a real number.*

Proof. Observe that
$$z\,\overline{z} = (a+ib)(a-ib) = a^2 + b^2 = |z|^2.$$
Since $|z|^2 = z\,\overline{z}$ is necessarily real and non-negative we can take square roots to obtain
$$\sqrt{z\,\overline{z}} = |z| = \sqrt{a^2+b^2} \in \mathbb{R}$$
as required. $\square$

Proposition 2.3. *If z_1, $z_2 \in \mathbb{C}$, then $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$.*

Proof. Let $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ so that
$$z_1 z_2 = (a_1 + ib_1)(a_2 + ib_2) = (a_1 a_2 - b_1 b_2) + i(b_1 a_2 + a_1 b_2)$$
implying that
$$\overline{z_1 z_2} = (a_1 a_2 - b_1 b_2) - i(b_1 a_2 + a_1 b_2).$$
On the other hand,
$$\overline{z_1}\,\overline{z_2} = (a_1 - ib_1)(a_2 - ib_2) = a_1 a_2 - b_1 b_2 - ib_1 a_2 - ia_1 b_2 = (a_1 a_2 - b_1 b_2) - i(b_1 a_2 + a_1 b_2)$$
as well, and the proof is complete. $\square$

Exercise 2.4. Let z_1, $z_2 \in \mathbb{C}$. Show that $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$.

Exercise 2.5. Let $z \in \mathbb{C}$. Show that $\overline{(\overline{z})} = z$.

Before proving the next proposition, we observe the geometric interpretation of $|z|$, $\text{Re}(z)$, and $\text{Im}(z)$ as shown in Figure 2.3 below.

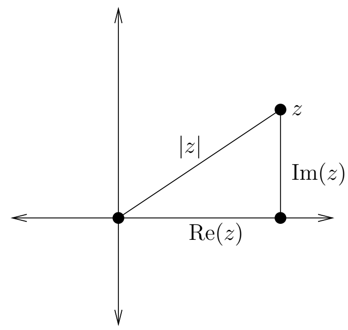

Figure 2.3: Geometric interpretation of $|z|$, $\text{Re}(z)$, and $\text{Im}(z)$.

Proposition 2.6. *If $z \in \mathbb{C}$, then*

(a) $\text{Re}(z) = \frac{1}{2}(z + \overline{z})$,

(b) $\text{Im}(z) = \frac{1}{2i}(z - \overline{z})$,

(c) $\text{Re}(z) \leq |z|$, *and*

(d) $\text{Im}(z) \leq |z|$.

Proof. Let $z = a + ib$ so that $\overline{z} = a - ib$. Solving the system of equations

$$z = a + ib \quad \text{and} \quad \overline{z} = a - ib$$

for a and b gives

$$a = \frac{1}{2}(z + \overline{z}) \quad \text{and} \quad b = \frac{1}{2i}(z - \overline{z}).$$

Moreover, since $|z| = \sqrt{a^2 + b^2}$, we see that

$$\text{Re}(z) = a \leq \sqrt{a^2 + b^2} = |z| \quad \text{and} \quad \text{Im}(z) = b \leq \sqrt{a^2 + b^2} = |z|$$

as required. □

Proposition 2.7. *If $z \in \mathbb{C}$, then $|\overline{z}| = |z|$.*

Proof. Let $z = a + ib$ so that $\overline{z} = a - ib$. Note that

$$|\overline{z}| = \sqrt{a^2 + (-b)^2} = \sqrt{a^2 + b^2} = |z|$$

as required. □

Geometrically this proposition says that length doesn't change under a reflection through the real axis; see Figure 2.4.

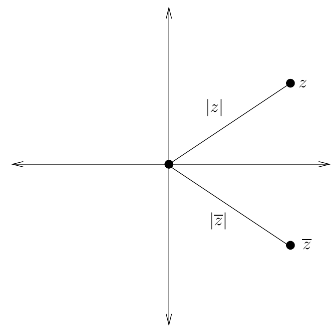

Figure 2.4: Geometric interpretation of $|\overline{z}| = |z|$.

Proposition 2.8. *If z_1, z_2 are complex variables, then $|z_1 z_2| = |z_1||z_2|$.*

Proof. Recall that $|w|^2 = w\,\overline{w}$ for any $w \in \mathbb{C}$. Let $w = z_1 z_2$ so that

$$|z_1 z_2|^2 = (z_1 z_2)(\overline{z_1 z_2}) = z_1 z_2(\overline{z_1}\,\overline{z_2}) = z_1(z_2\,\overline{z_1})\overline{z_2} = z_1(\overline{z_1}\,z_2)\overline{z_2} = (z_1\,\overline{z_1})(z_2\,\overline{z_2}) = |z_1|^2|z_2|^2$$

using Proposition 2.3, the Associative Law twice, and the Commutative Law. Since the moduli in question are non-negative real numbers we can take square roots to obtain

$$|z_1 z_2| = |z_1||z_2|$$

as required. □

Proposition 2.9. *If z_1, z_2 are complex variables with $z_2 \neq 0$, then*

$$\frac{z_1}{z_2} = \frac{z_1\,\overline{z_2}}{|z_2|^2}.$$

Proof. Observe that

$$\frac{z_1}{z_2} = \frac{z_1}{z_2}\,\frac{\overline{z_2}}{\overline{z_2}} = \frac{z_1\,\overline{z_2}}{|z_2|^2}$$

as required. □

Theorem 2.10 (Triangle Inequality)**.** *If z_1, z_2 are complex variables, then*

$$|z_1 + z_2| \leq |z_1| + |z_2|.$$

Proof. Recall that $|w|^2 = w\,\overline{w}$ for any $w \in \mathbb{C}$. Taking $w = z_1 + z_2$ implies

$$\begin{aligned} |z_1 + z_2|^2 = (z_1 + z_2)(\overline{z_1 + z_2}) = (z_1 + z_2)(\overline{z_1} + \overline{z_2}) &= z_1\,\overline{z_1} + z_2\,\overline{z_1} + z_1\,\overline{z_2} + z_2\,\overline{z_2} \\ &= |z_1|^2 + |z_2|^2 + z_2\,\overline{z_1} + z_1\,\overline{z_2} \end{aligned}$$

using Exercise 2.4 and the Distributive Law. The next step is to deal with $z_2\,\overline{z_1} + z_1\,\overline{z_2}$. Recall that $2\,\mathrm{Re}(w) = w + \overline{w}$. If we take $w = z_1\,\overline{z_2}$, then

$$\overline{w} = \overline{z_1\,\overline{z_2}} = \overline{z_1}\,\overline{(\overline{z_2})} = \overline{z_1}\,z_2$$

using Proposition 2.3 and Exercise 2.5, and so we see that

$$z_2\,\overline{z_1} + z_1\,\overline{z_2} = \overline{w} + w = 2\,\mathrm{Re}(w) = 2\,\mathrm{Re}(z_1\,\overline{z_2}).$$

However, we also know from Proposition 2.6 that $\mathrm{Re}(w) \leq |w|$ which implies that

$$\mathrm{Re}(z_1\,\overline{z_2}) \leq |z_1\,\overline{z_2}| = |z_1||\overline{z_2}| = |z_1||z_2|.$$

Therefore, we conclude that

$$|z_1 + z_2|^2 \leq |z_1|^2 + |z_2|^2 + 2|z_1||z_2| = (|z_1| + |z_2|)^2.$$

Since both sides of the inequality involve only non-negative real numbers, we can take square roots to obtain

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

as required. □