Lecture #28: Laurent Series

Recall from Lecture #27 that we considered the function

$$f(z) = \frac{1+2z}{z^2+z^3}$$

and we formally manipulated $f(z)$ to obtain the infinite expansion

$$f(z) = \frac{1}{z^2} + \frac{1}{z} - 1 + z - z^2 + \cdots .$$

Observe that $f(z)$ is analytic in the annulus $0 < |z| < 1$. Does

$$\frac{1+2z}{z^2+z^3} = \frac{1}{z^2} + \frac{1}{z} - 1 + z - z^2 + \cdots$$

for all $0 < |z| < 1$? The answer turns out to be yes. Thus, our goal for today is to prove that if a function $f(z)$ is analytic in an annulus, then it has an infinite series expansion which converges for all z in the annulus. This expansion is known as the *Laurent series for $f(z)$*.

Theorem 28.1. *Suppose that $f(z)$ is analytic in the annulus $r < |z - z_0| < R$ (with $r = 0$ and $R = \infty$ allowed). Then $f(z)$ can be represented as*

$$f(z) = \sum_{j=0}^{\infty} a_j (z-z_0)^j + \sum_{j=1}^{\infty} a_{-j}(z-z_0)^{-j} \qquad (*)$$

where

$$a_j = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z_0)^{j+1}}\, \mathrm{d}\zeta, \quad j = 0, \pm 1, \pm 2, \ldots,$$

and C is any closed contour oriented counterclockwise that surrounds z_0 and lies entirely in the annulus.

The proof is very similar to the proof of Theorem 26.1 for the Taylor series representation of an analytic function in a disk $|z - z_0| < R$. We will not include the full proof, but instead give an indication of where the formula for a_j comes from. Suppose that $f(z)$ can be represented as

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z-z_0)^k$$

with convergence in the annulus $r < |z - z_0| < R$. Observe that

$$\frac{1}{2\pi i} f(z)(z-z_0)^{-j} = \sum_{k=-\infty}^{\infty} \frac{a_k}{2\pi i}(z-z_0)^{k-j}$$

and so

$$\frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^j}\,\mathrm{d}z = \sum_{k=-\infty}^{\infty} \frac{a_k}{2\pi i}\int_C \frac{1}{(z-z_0)^{j-k}}\,\mathrm{d}z$$

where C is any closed contour oriented counterclockwise that surrounds z_0 and lies entirely in the annulus. We now observe from Theorem 23.2 that

$$\int_C \frac{1}{(z-z_0)^{j-k}}\,\mathrm{d}z = 2\pi i \quad \text{if} \quad k = j-1$$

and

$$\int_C \frac{1}{(z-z_0)^{j-k}}\,\mathrm{d}z = 0 \quad \text{if} \quad k \neq j-1$$

so that

$$\frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^j}\,\mathrm{d}z = \sum_{k=-\infty}^{\infty} \frac{a_k}{2\pi i}\int_C \frac{1}{(z-z_0)^{j-k}}\,\mathrm{d}z = a_{j-1}.$$

In other words,

$$a_{j-1} = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^j}\,\mathrm{d}z \quad \text{or, equivalently,} \quad a_j = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{j+1}}\,\mathrm{d}z.$$

Remark. Observe that

$$a_j = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta-z_0)^{j+1}}\,\mathrm{d}\zeta$$

so, at least for $j = 0, 1, 2, \ldots$, it might be tempting to use the Cauchy Integral Formula (Theorem 25.4) to try and conclude that a_j is equal to

$$\frac{f^{(j)}(z_0)}{j!}$$

as was the case in the Taylor series derivation. This is not true, however, since the assumption on $f(z)$ is that it is analytic in the annulus $r < |z-z_0| < R$. This means that if C is a closed contour oriented counterclockwise lying in the annulus and surrounding z_0, there is no guarantee that $f(z)$ is analytic everywhere inside C which is the assumption required in order to apply the Cauchy Integral Formula. Thus, although there is a seemingly simple formula for the coefficients a_j in the Laurent series expansion, the computation of a_j as a contour integral is not necessarily a straightforward application of the Cauchy Integral Formula.

Example 28.2. Suppose that

$$f(z) = \frac{1+2z}{z^2+z^3}$$

which is analytic for $0 < |z| < 1$. Show that the Laurent series expansion of $f(z)$ for $0 < |z| < 1$ is

$$\frac{1}{z^2} + \frac{1}{z} - 1 + z - z^2 + \cdots .$$

Solution. Suppose that C is any closed contour oriented counterclockwise lying entirely in $\{0 < |z| < 1\}$ and surrounding $z_0 = 0$. Consider

$$a_j = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z_0)^{j+1}}\,\mathrm{d}\zeta = \frac{1}{2\pi i}\int_C \frac{1+2\zeta}{\zeta^2+\zeta^3}\cdot\frac{1}{\zeta^{j+1}}\,\mathrm{d}\zeta = \frac{1}{2\pi i}\int_C \frac{(1+2\zeta)/(1+\zeta)}{\zeta^{j+3}}\,\mathrm{d}\zeta.$$

The reason for writing it in this form is that now we can apply the Cauchy Integral Formula to compute

$$\frac{1}{2\pi i}\int_C \frac{(1+2\zeta)/(1+\zeta)}{\zeta^{j+3}}\,\mathrm{d}\zeta.$$

Observe that the function

$$g(z) = \frac{1+2z}{1+z}$$

is analytic inside and on C. Thus, the Cauchy Integral Theorem implies that if $j \le -3$, then

$$\frac{1}{2\pi i}\int_C \frac{(1+2\zeta)/(1+\zeta)}{\zeta^{j+3}}\,\mathrm{d}\zeta = 0$$

so that $a_{-3} = a_{-4} = \cdots = 0$. The Cauchy Integral Formula implies that if $j \ge -2$, then

$$\frac{1}{2\pi i}\int_C \frac{(1+2\zeta)/(1+\zeta)}{\zeta^{j+3}}\,\mathrm{d}\zeta = \frac{1}{2\pi i}\int_C \frac{g(\zeta)}{\zeta^{j+3}}\,\mathrm{d}\zeta = \frac{g^{(j+2)}(0)}{(j+2)!}.$$

Note that if $j = -2$, then $a_{-2} = g(0) = 1$. In order to compute successive derivatives of $g(z)$, observe that

$$g(z) = \frac{1+2z}{1+z} = \frac{1}{1+z} + \frac{2z}{1+z}$$

Now, if $k = 1, 2, 3, \ldots$, then

$$\frac{\mathrm{d}^k}{\mathrm{d}z^k}\left(\frac{1}{1+z}\right) = (-1)^k \frac{k!}{(1+z)^{k+1}}$$

and

$$\frac{\mathrm{d}^k}{\mathrm{d}z^k}\left(\frac{z}{1+z}\right) = (-1)^{k+1}\frac{k!}{(1+z)^k} + (-1)^k \frac{k!z}{(1+z)^{k+1}}$$

so that

$$g^{(k)}(0) = (-1)^k k! + 2(-1)^{k+1}k! = (-1)^{k+1}k! \quad \text{for } k = 1, 2, 3, \ldots.$$

This implies

$$a_j = \frac{g^{(j+2)}(0)}{(j+2)!} = \frac{(-1)^{j+3}(j+2)!}{(j+2)!} = (-1)^{j+3} = (-1)^{j+1} \quad \text{for } j = -1, 0, 1, 2\ldots$$

so that the Laurent series expansion of $f(z)$ for $0 < |z| < 1$ is

$$\frac{1}{z^2} + \frac{1}{z} - 1 + z - z^2 + \cdots = \frac{1}{z^2} + \sum_{j=-1}^{\infty}(-1)^{j+1}z^j.$$

Remark. We will soon learn other methods for determining Laurent series expansions that do not require the computation of contour integrals.