Mathematics 312 (Fall 2013) November 14, 2013
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Lecture #27: Taylor Series and Isolated Singularities

Recall from last class that if $f(z)$ is analytic at a, then

$$f(z) = \sum_{j=0}^{\infty} \frac{f^j(a)}{j!}(z-a)^j$$

for all z in some neighbourhood of a. This neighbourhood, denoted by $\{|z-a| < R\}$, is called the *disk of convergence of the Taylor series for* $f(z)$.

Example 27.1. We know from Lecture #6 that

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots$$

for $|z| < 1$. Since $(1-z)^{-1}$ is analytic for $|z| < 1$, we conclude this must be its Taylor series expansion about $a = 0$. Moreover, since $|-z^2| < 1$ whenever $|z| < 1$, we find

$$\frac{1}{1+z^2} = 1 + (-z^2) + (-z^2)^2 + (-z^2)^3 + \cdots = 1 - z^2 + z^4 - z^6 + z^8 - z^{10} + \cdots$$

for $|z| < 1$. Observe now that

$$\frac{\mathrm{d}}{\mathrm{d}z} \arctan z = \frac{1}{1+z^2}.$$

Consequently we can use Theorem 26.4, which tells us that the derivative of an analytic function $f(z)$ can be obtained by termwise differentiation of the Taylor series of $f(z)$, to conclude that the Taylor series expansion about 0 for $\arctan z$ must be

$$\arctan z = z - \frac{z^3}{3} + \frac{z^5}{5} - \frac{z^7}{7} + \cdots$$

for $|z| < 1$ since

$$\frac{\mathrm{d}}{\mathrm{d}z} \arctan z = \frac{\mathrm{d}}{\mathrm{d}z}\left(z - \frac{z^3}{3} + \frac{z^5}{5} - \frac{z^7}{7} + \cdots\right) = 1 - z^2 + z^4 - z^6 + z^8 - z^{10} + \cdots = \frac{1}{1+z^2}.$$

We can now recover the famous Leibniz formula for π from 1682; that is, since $\arctan(1) = \pi/4$, we find

$$\frac{\pi}{4} = \arctan(1) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\,,$$

or equivalently,

$$\pi = 4\sum_{j=0}^{\infty} \frac{(-1)^j}{2j+1}.$$

Observe that in the last example we obtained the Taylor series for

$$\frac{1}{1+z^2}$$

by formally plugging $-z^2$ into the Taylor series for

$$\frac{1}{1-z}.$$

Suppose that we try do the same thing with a point at which the Taylor series is not analytic. For instance, we know that

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

for all $z \in \mathbb{C}$. However, is it true that

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2!} + \frac{1}{3!z^3} + \cdots ,$$

at least for $z \neq 0$? Our goal is to now develop the theory of Laurent series which will provide us with the means to understand the series expansion for $e^{1/z}$ given above.

Consider again the function $e^{1/z}$. Observe that the point $z = 0$ is special because it is the only point at which $e^{1/z}$ fails to be analytic. We call such a point an *isolated singularity.*

Definition. A point z_0 is called an *isolated singular point* of $f(z)$ if $f(z)$ is not analytic at z_0 but is analytic at all points in some small neighbourhood of z_0.

Example 27.2. If $f(z) = e^{1/z}$, then $z = 0$ is an isolated singular point of $f(z)$.

Example 27.3. If

$$f(z) = \frac{1}{z},$$

then $z_0 = 0$ is an isolated singular point of $f(z)$.

Example 27.4. If

$$f(z) = \frac{e^z}{\cos z},$$

then $z_0 = \pi/2$ is an isolated singular point of $f(z)$. Moreover, since $\cos z_0 = 0$ if and only if $z_0 \in \{\pi/2 + \pi k : k \in \mathbb{Z}\}$, we conclude that $f(z)$ has infinitely many isolated singular points, namely $\{\pi/2 + \pi k : k \in \mathbb{Z}\}$.

Example 27.5. If

$$f(z) = \csc z = \frac{1}{\sin z},$$

then $f(z)$ has isolated singular points at $z_0 \in \{n\pi : n \in \mathbb{Z}\}$.

Example 27.6. If

$$f(z) = \frac{1}{\sin(1/z)},$$

then $f(z)$ has isolated singular points at those points for which $\sin(1/z) = 0$, namely

$$\frac{1}{z} = n\pi \quad \text{or equivalently} \quad z = \frac{1}{n\pi}$$

for $n = \pm 1, \pm 2, \pm 3, \ldots$. Note that $z = 0$ is not an isolated singular point since there are points of the form $1/(n\pi)$ arbitrarily close to 0 for n sufficiently large. In other words, 0 is a cluster point or an accumulation point of the sequence of isolated singular points $1/(n\pi)$, $n = 1, -1, 2, -2, 3, -3, \ldots$

The basic idea is as follows. For a function $f(z)$ analytic for $|z - z_0| < R$, we have

$$f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j.$$

However, if a function is analytic in an annulus only, say $r < |z - z_0| < R$ (with $r = 0$ and $R = \infty$ allowed), then our expansion will have negative powers of $(z - z_0)$; for example,

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2!} + \frac{1}{3!z^3} + \cdots .$$

Example 27.7. Determine a series expansion for

$$f(z) = \frac{1 + 2z}{z^2 + z^3}$$

in powers of z.

Solution. Recall that

$$\frac{1}{1+z} = 1 - z + z^2 - z^3 + \cdots .$$

Hence,

$$\begin{aligned} f(z) = \frac{1+2z}{z^2(1+z)} = \frac{1+2z}{z^2}\frac{1}{1+z} &= \frac{1+2z}{z^2}\left(1 - z + z^2 - z^3 + \cdots\right) \\ &= \frac{1}{z^2}\left[\left(1 - z + z^2 - z^3 + \cdots\right) + 2z\left(1 - z + z^2 - z^3 + \cdots\right)\right] \\ &= \frac{1}{z^2}\left(1 + z - z^2 + z^3 - z^4 + \cdots\right) \\ &= \frac{1}{z^2} + \frac{1}{z} - 1 + z - z^2 + \cdots . \end{aligned}$$

This expansion is the so-called Laurent series expansion of $f(z)$ about $z_0 = 0$. The next several lectures will be devoted to the development of the theory of Laurent series. Here is one important use of the series expansion just determined.

Example 27.8. Suppose that $C = \{|z| = 1/2\}$ oriented counterclockwise. Compute

$$\int_C \frac{1+2z}{z^2+z^3}\,\mathrm{d}z.$$

Solution. Assuming that

$$\frac{1+2z}{z^2+z^3} = \frac{1}{z^2} + \frac{1}{z} - 1 + z - z^2 + \cdots\,,$$

we obtain

$$\begin{aligned}\int_C \frac{1+2z}{z^2+z^3}\,\mathrm{d}z &= \int_C \left(\frac{1}{z^2} + \frac{1}{z} - 1 + z - z^2 + \cdots\right)\mathrm{d}z \\ &= \int_C \frac{1}{z^2}\,\mathrm{d}z + \int_C \frac{1}{z}\,\mathrm{d}z - \int_C 1\,\mathrm{d}z + \int_C z\,\mathrm{d}z - \int_C z^2\,\mathrm{d}z + \cdots \\ &= 0 + 2\pi i + 0 + 0 + 0 + \cdots \\ &= 2\pi i.\end{aligned}$$