Lecture #25: Consequences of the Cauchy Integral Formula

The main result that we will establish today is that an analytic function has derivatives of all orders. The key to establishing this is to first prove a slightly more general result.

Theorem 25.1. *Let g be continuous on the contour C and for each z_0 not on C, set*

$$G(z_0) = \int_C \frac{g(\zeta)}{\zeta - z_0}\, \mathrm{d}\zeta.$$

Then G is analytic at z_0 with

$$G'(z_0) = \int_C \frac{g(\zeta)}{(\zeta - z_0)^2}\, \mathrm{d}\zeta. \tag{$*$}$$

Remark. Observe that in the statement of the theorem, we do not need to assume that g is analytic or that C is a closed contour.

Proof. Let z_0 not on C be fixed. In order to prove the differentiability of G and the desired formula for $G'(z_0)$, we must show that

$$\lim_{\Delta z \to 0} \frac{G(z_0 + \Delta z) - G(z_0)}{\Delta z} = \int_C \frac{g(\zeta)}{(\zeta - z_0)^2}\, \mathrm{d}\zeta.$$

Observe that

$$\begin{aligned}
\frac{G(z_0 + \Delta z) - G(z_0)}{\Delta z} &= \frac{1}{\Delta z} \int_C \frac{g(\zeta)}{\zeta - (z_0 + \Delta z)} - \frac{g(\zeta)}{\zeta - z_0}\, \mathrm{d}\zeta \\
&= \frac{1}{\Delta z} \int_C g(\zeta) \left[\frac{1}{\zeta - (z_0 + \Delta z)} - \frac{1}{\zeta - z_0} \right] \mathrm{d}\zeta \\
&= \int_C \frac{g(\zeta)}{(\zeta - z_0 - \Delta z)(\zeta - z_0)}\, \mathrm{d}\zeta
\end{aligned}$$

and so (with a bit of algebra)

$$\begin{aligned}
\frac{G(z_0 + \Delta z) - G(z_0)}{\Delta z} - \int_C \frac{g(\zeta)}{(\zeta - z_0)^2}\, \mathrm{d}\zeta &= \int_C \frac{g(\zeta)}{(\zeta - z_0 - \Delta z)(\zeta - z_0)}\, \mathrm{d}\zeta - \int_C \frac{g(\zeta)}{(\zeta - z_0)^2}\, \mathrm{d}\zeta \\
&= \Delta z \int_C \frac{g(\zeta)}{(\zeta - z_0 - \Delta z)(\zeta - z_0)^2}\, \mathrm{d}\zeta
\end{aligned} \tag{$\dagger$}$$

The next step is to show that

$$\left| \int_C \frac{g(\zeta)}{(\zeta - z_0 - \Delta z)(\zeta - z_0)^2}\, \mathrm{d}\zeta \right|$$

is bounded.

To this end, let $M = \max\{|g(\zeta)| : \zeta \in C\}$ be the maximum value of $|g(\zeta)|$ on C, and let $d = \min\{\text{dist}(z_0, w) : w \in C\}$ be the minimal distance from z_0 to C. Note that $|\zeta - z_0| \geq d > 0$ for all ζ on C. Without loss of generality, assume that $|\Delta z| < d/2$ (since we ultimately care about $\Delta z \to 0$, this is a valid assumption). By the triangle inequality, if $\zeta \in C$, then

$$|\zeta - z_0 - \Delta z| \geq |\zeta - z_0| - |\Delta z| \geq d - \frac{d}{2} = \frac{d}{2},$$

and so

$$\begin{aligned}\left|\int_C \frac{g(\zeta)}{(\zeta - z_0 - \Delta z)(\zeta - z_0)^2}\,\mathrm{d}\zeta\right| \leq \int_C \left|\frac{g(\zeta)}{(\zeta - z_0 - \Delta z)(\zeta - z_0)^2}\right|\,|\mathrm{d}\zeta| &\leq \frac{M}{\frac{d}{2}\cdot d^2}\int_C 1\,|\mathrm{d}\zeta| \\ &= \frac{2M\ell(C)}{d^3}\end{aligned}$$

where $\ell(C) < \infty$ is the arclength of the contour C. Hence, considering (†), we find

$$\begin{aligned}\lim_{\Delta z \to 0}\left|\frac{G(z_0 + \Delta z) - G(z_0)}{\Delta z} - \int_C \frac{g(\zeta)}{(\zeta - z_0)^2}\,\mathrm{d}\zeta\right| &= \lim_{\Delta z \to 0}\left|\Delta z\int_C \frac{g(\zeta)}{(\zeta - z_0 - \Delta z)(\zeta - z_0)^2}\,\mathrm{d}\zeta\right| \\ &\leq \frac{2M\ell(C)}{d^3}\lim_{\Delta z \to 0}|\Delta z| \\ &= 0\end{aligned}$$

so that $(*)$ holds. Note that we have proved $G(z_0)$ is differentiable at $z_0 \notin C$ for z_0 fixed. Since z_0 was arbitrary, we conclude that $G(z_0)$ is differentiable at *any* $z_0 \notin C$ implying that G is analytic at $z_0 \notin C$ as required. □

It is important to note that exactly the same method of proof yields the following result.

Corollary 25.2. *Let g be continuous on the contour C and for each z_0 not on C, set*

$$H(z_0) = \int_C \frac{g(\zeta)}{(\zeta - z_0)^n}\,\mathrm{d}\zeta$$

where n is a positive integer. Then H is analytic at z_0 with

$$H'(z_0) = n\int_C \frac{g(\zeta)}{(\zeta - z_0)^{n+1}}\,\mathrm{d}\zeta. \qquad (**)$$

Now we make a very important observation that follows immediately from Theorem 25.1 and Corollary 25.2.

Theorem 25.3. *If $f(z)$ is analytic in a domain D, then all of its derivatives $f'(z)$, $f''(z)$, $f'''(z)$, … exist and are themselves analytic.*

Remark. This theorem is remarkable because it is unique to complex analysis. The analogue for real-valued functions is not true. For example, $f(x) = 9x^{5/3}$ for $x \in \mathbb{R}$ is differentiable for all x, but its derivative $f'(x) = 15x^{2/3}$ is not differentiable at $x = 0$ (i.e., $f''(x) = 10x^{-1/3}$ does not exist when $x = 0$).

Moreover, if the function in the statement of Theorem 25.1 happens to be analytic and C happens to be a closed contour oriented counterclockwise, then we arrive at the following important theorem which might be called the General Version of the Cauchy Integral Formula.

Theorem 25.4 (Cauchy Integral Formula, General Version)**.** *Suppose that $f(z)$ is analytic inside and on a simply closed contour C oriented counterclockwise. If z is any point inside C, then*

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} \, \mathrm{d}\zeta,$$

$n = 1, 2, 3, \ldots$.

For the purposes of computations, it is usually more convenient to write the General Version of the Cauchy Integral Formula as follows.

Corollary 25.5. *Suppose that $f(z)$ is analytic inside and on a simply closed contour C oriented counterclockwise. If a is any point inside C, then*

$$\int_C \frac{f(z)}{(z-a)^m} \, \mathrm{d}z = \frac{2\pi i f^{(m-1)}(a)}{(m-1)!}.$$

Example 25.6. Compute

$$\int_C \frac{e^{5z}}{(z-i)^3} \, \mathrm{d}z$$

where $C = \{|z| = 2\}$ is the circle of radius 2 centred at 0 oriented counterclockwise.

Solution. Let $f(z) = e^{5z}$ so that $f(z)$ is entire, and let $a = i$ which is inside C. Therefore,

$$\int_C \frac{e^{5z}}{(z-i)^3} \, \mathrm{d}z = \frac{2\pi i f''(i)}{2!} = 25\pi i e^{5i}$$

since $f''(z) = 25e^{5z}$.