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Prof. Michael Kozdron

Lecture #24: The Cauchy Integral Formula

Recall that the Cauchy Integral Theorem, Basic Version states that if D is a domain and $f(z)$ is analytic in D with $f'(z)$ continuous, then

$$\int_C f(z)\,\mathrm{d}z = 0$$

for any closed contour C lying entirely in D having the property that C is continuously deformable to a point.

We also showed that if C is any closed contour oriented counterclockwise in $\mathbb{C}$ and a is inside C, then

$$\int_C \frac{1}{z-a}\,\mathrm{d}z = 2\pi i. \qquad (*)$$

Our goal now is to derive the celebrated Cauchy Integral Formula which can be viewed as a generalization of $(*)$.

Theorem 24.1 (Cauchy Integral Formula)**.** *Suppose that D is a domain and that $f(z)$ is analytic in D with $f'(z)$ continuous. If C is a closed contour oriented counterclockwise lying entirely in D having the property that the region surrounded by C is a simply connected subdomain of D (i.e., if C is continuously deformable to a point) and a is inside C, then*

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,\mathrm{d}z.$$

Proof. Observe that we can write

$$\int_C \frac{f(z)}{z-a}\,\mathrm{d}z = \int_C \frac{f(a)}{z-a}\,\mathrm{d}z + \int_C \frac{f(z)-f(a)}{z-a}\,\mathrm{d}z = 2\pi f(a) i + \int_{C_a} \frac{f(z)-f(a)}{z-a}\,\mathrm{d}z$$

where $C_a = \{|z-a| = r\}$ oriented counterclockwise since $(*)$ implies

$$\int_C \frac{f(a)}{z-a}\,\mathrm{d}z = f(a)\int_C \frac{1}{z-a}\,\mathrm{d}z = 2\pi f(a) i$$

and

$$\int_C \frac{f(z)-f(a)}{z-a}\,\mathrm{d}z = \int_{C_a} \frac{f(z)-f(a)}{z-a}\,\mathrm{d}z$$

since the integrand

$$\frac{f(z)-f(a)}{z-a}$$

is analytic everywhere except at $z=a$ and its derivative is continuous everywhere except at $z=a$ so that integration over C can be continuously deformed to integration over C_a. However, if we write

$$\int_C \frac{f(z)}{z-a}\,\mathrm{d}z - 2\pi f(a) i = \int_{C_a} \frac{f(z)-f(a)}{z-a}\,\mathrm{d}z,$$

and note that the left side of the previous expression does not depend on r, then we conclude

$$\int_C \frac{f(z)}{z-a}\,\mathrm{d}z - 2\pi f(a) i = \lim_{r\downarrow 0} \int_{C_a} \frac{f(z)-f(a)}{z-a}\,\mathrm{d}z.$$

Hence, the proof will be complete if we can show that

$$\lim_{r\downarrow 0} \int_{C_a} \frac{f(z)-f(a)}{z-a}\,\mathrm{d}z = 0.$$

To this end, suppose that $M_r = \max\{|f(z)-f(a)|,\ z \text{ on } C_a\}$. Therefore, if z is on $C_a = \{|z-a| = r\}$, then

$$\left|\frac{f(z)-f(a)}{z-a}\right| = \frac{|f(z)-f(a)|}{|z-a|} = \frac{|f(z)-f(a)|}{r} \le \frac{M_r}{r}$$

so that

$$\begin{aligned}\left|\int_{C_a} \frac{f(z)-f(a)}{z-a}\,\mathrm{d}z\right| \le \int_{C_a} \left|\frac{f(z)-f(a)}{z-a}\right| |\mathrm{d}z| \le \int_{C_a} \frac{M_r}{r}\,|\mathrm{d}z| = \frac{M_r}{r}\int_{C_a} 1\,|\mathrm{d}z| &= \frac{M_r}{r}\ell(C_a) \\ &= \frac{M_r}{r}\cdot 2\pi r \\ &= 2\pi M_r\end{aligned}$$

since the arclength of C_a is $\ell(C_a) = 2\pi r$. However, since $f(z)$ is analytic in D, we know that $f(z)$ is necessarily continuous in D so that

$$\lim_{z\to a} |f(z)-f(a)| = 0 \quad \text{or, equivalently,} \quad \lim_{r\downarrow 0} M_r = 0.$$

Therefore,

$$\lim_{r\downarrow 0}\left|\int_{C_a} \frac{f(z)-f(a)}{z-a}\,\mathrm{d}z\right| \le \lim_{r\downarrow 0}(2\pi M_r) = 0$$

as required. □

Example 24.2. Compute

$$\frac{1}{2\pi i}\int_C \frac{ze^z}{z-i}\,\mathrm{d}z$$

where $C = \{|z| = 2\}$ is the circle of radius 2 centred at 0 oriented counterclockwise.

Solution. Observe that $f(z) = ze^z$ is entire, $f'(z) = ze^z + e^z$ is continuous, and i is inside C. Therefore, by the Cauchy Integral Formula,

$$\frac{1}{2\pi i}\int_C \frac{ze^z}{z-i}\,\mathrm{d}z = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-i}\,\mathrm{d}z = f(i) = ie^i.$$

Example 24.3. Compute

$$\int_C \frac{ze^z}{z+i}\,\mathrm{d}z$$

where $C = \{|z| = 2\}$ is the circle of radius 2 centred at 0 oriented counterclockwise.

Solution. Observe that $f(z) = ze^z$ is entire, $f'(z) = ze^z + e^z$ is continuous, and $-i$ is inside C. Therefore, by the Cauchy Integral Formula,

$$\int_C \frac{ze^z}{z+i}\,\mathrm{d}z = \int_C \frac{f(z)}{z+i}\,\mathrm{d}z = 2\pi i f(-i) = 2\pi i \cdot -ie^{-i} = 2\pi e^{-i}.$$

Example 24.4. Compute

$$\int_C \frac{ze^z}{z^2+1}\,\mathrm{d}z$$

where $C = \{|z| = 2\}$ is the circle of radius 2 centred at 0 oriented counterclockwise.

Solution. Observe that partial fractions implies

$$\frac{1}{z^2+1} = \frac{1}{z^2-i^2} = \frac{1}{(z+i)(z-i)} = \frac{i/2}{z+i} - \frac{i/2}{z-i}$$

and so

$$\int_C \frac{ze^z}{z^2+1}\,\mathrm{d}z = \frac{i}{2}\int_C \frac{ze^z}{z+i}\,\mathrm{d}z - \frac{i}{2}\int_C \frac{ze^z}{z-i}\,\mathrm{d}z.$$

Let $f(z) = ze^z$. Note that $f(z)$ is entire and $f'(z) = ze^z + e^z$ is continuous. Since both i and $-i$ are inside C, the Cauchy Integral Formula implies

$$\int_C \frac{ze^z}{z+i}\,\mathrm{d}z = 2\pi i f(-i) = 2\pi i \cdot -ie^{-i} = 2\pi e^{-i} \text{ and } \int_C \frac{ze^z}{z-i}\,\mathrm{d}z = 2\pi i f(i) = 2\pi i \cdot ie^{i} = -2\pi e^{i}$$

so that

$$\int_C \frac{ze^z}{z^2+1}\,\mathrm{d}z = \frac{i}{2}\cdot 2\pi e^{-i} - \frac{i}{2}\cdot -2\pi e^{i} = \pi i e^{-i} + \pi i e^{i} = 2\pi i \left[\frac{e^{i}+e^{-i}}{2}\right] = 2\pi i \cos 1.$$