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Lecture #22: Applications of the Cauchy Integral Theorem

Example 22.1. Suppose $p \in \mathbb{Z}$. Compute

$$\int_C z^p \, \mathrm{d}z$$

where $C = \{|z| = r\}$ is the circle of radius $r > 0$ centred at 0 oriented counterclockwise.

Solution. We will consider separately two cases, namely (i) $p = 0, 1, 2, \ldots$, and (ii) $p = -1, -2, \ldots$.

In the first case, we can use either the Fundamental Theorem of Calculus for Integrals over Closed Contours or the Cauchy Integral Theorem to conclude

$$\int_C z^p \, \mathrm{d}z = 0.$$

To use the FTC, observe that $f(z) = z^p$, $p = 0, 1, 2, \ldots$, is continuous in $\mathbb{C}$ and that $F(z) = (p+1)^{-1} z^{p+1}$ is analytic in $\mathbb{C}$ with $F'(z) = z^p = f(z)$. Since C is a closed contour, the hypotheses of the FTC have been met so that

$$\int_C z^p \, \mathrm{d}z = 0.$$

Alternatively, the function $f(z) = z^p$, $p = 0, 1, 2, \ldots$, is analytic in $\mathbb{C}$ with $f'(z) = 0$ for $p = 0$ and $f'(z) = pz^{p-1}$ for $p = 1, 2, \ldots$. In any case, $f'(z)$ is continuous in $\mathbb{C}$. Thus, the hypotheses of the Cauchy Integral Theorem, Basic Version have been met and so we conclude

$$\int_C z^p \, \mathrm{d}z = 0$$

as before.

In the second case, consider the function $f(z) = z^p$, $p = -1, -2, \ldots$. This function is not defined at $z = 0$ and so it is necessarily not continuous at $z = 0$ and not analytic at $z = 0$. Thus, in order to compute

$$\int_C z^p \, \mathrm{d}z$$

we cannot use either the FTC or the Cauchy Integral Theorem. Hence, we must compute it as a contour integral using a parametrization of C. Let $z(t) = re^{it}$, $0 \le t \le 2\pi$, parametrize C oriented counterclockwise. Since $z'(t) = ire^{it}$, we find

$$\int_C z^p \, \mathrm{d}z = \int_0^{2\pi} (re^{it})^p \cdot ire^{it} \, \mathrm{d}t = ir^{p+1} \int_0^{2\pi} e^{i(p+1)t} \, \mathrm{d}t.$$

We now consider two subcases. If $p = -1$, then

$$\int_C z^p \, \mathrm{d}z = \int_C z^{-1} \, \mathrm{d}z = ir^{(-1)+1} \int_0^{2\pi} e^{i((-1)+1)t} \, \mathrm{d}t = i \int_0^{2\pi} \mathrm{d}t = 2\pi i.$$

If $p = -2, -3, \ldots$, then

$$\int_C z^p \, \mathrm{d}z = ir^{p+1} \int_0^{2\pi} e^{i(p+1)t} \, \mathrm{d}t = \frac{ir^{p+1}}{i(p+1)} e^{i(p+1)t} \Big|_{t=0}^{t=2\pi} = \frac{r^{p+1}}{p+1}[e^{i(p+1)2\pi} - 1] = 0$$

since $e^{i(p+1)2\pi} = 1$ as $p \in \mathbb{Z}$.

In summary, we have

$$\int_C z^p \, \mathrm{d}z = \begin{cases} 2\pi i, & \text{if } p = -1, \\ 0, & \text{if } p \in \mathbb{Z},\, p \neq 0. \end{cases}$$

Example 22.2. Suppose that $C_a = \{|z - a| < r\}$ denotes the circle of radius $r > 0$ centred at a oriented counterclockwise. Compute

$$\int_{C_a} \frac{1}{z - a} \, \mathrm{d}z.$$

Solution. Since the function

$$f(z) = \frac{1}{z - a}$$

is not analytic at a which happens to be inside C_a, we must evaluate this contour integral by definition. Let $z(t) = a + re^{it}$, $0 \leq t \leq 2\pi$, parametrize C so that $z'(t) = ire^{it}$. Therefore,

$$\int_{C_a} \frac{1}{z - a} \, \mathrm{d}z = \int_0^{2\pi} \frac{1}{z(t) - a} z'(t) \, \mathrm{d}t = \int_0^{2\pi} \frac{ire^{it}}{a + re^{it} - a} \, \mathrm{d}t = \int_0^{2\pi} i \, \mathrm{d}t = 2\pi i.$$

Example 22.3. Suppose that $C = \{|z| = r\}$ is the circle of radius $r > 0$ centred at 0 oriented counterclockwise. Let $a \in \mathbb{C}$. Compute

$$\int_C \frac{1}{z - a} \, \mathrm{d}z$$

assuming that (i) a is outside C and (ii) a is inside C. (Note that a is never *on* C.)

Solution. If a is outside C, then there is some simply connected domain D containing C but not a. Moreover, the function $f(z) = (z - a)^{-1}$ is analytic in D with $f'(z) = -(z - a)^{-2}$ for $z \in D$ so that $f'(z)$ is continuous in D. Hence, the hypotheses of the Cauchy Integral Theorem, Basic Version have been met so that

$$\int_C \frac{1}{z - a} \, \mathrm{d}z = 0.$$

On the other hand, suppose that a is inside C and let R denote the interior of C. Since the function $f(z) = (z - a)^{-1}$ is not analytic in any domain containing R, we cannot apply the Cauchy Integral Theorem. Hence, we must compute it as a contour integral using a parametrization of C. Let $z(t) = re^{it}$, $0 \leq t \leq 2\pi$, parametrize C oriented counterclockwise. Since $z'(t) = ire^{it}$, we find

$$\int_C \frac{1}{z - a} \, \mathrm{d}z = \int_0^{2\pi} \frac{1}{re^{it} - a} \cdot ire^{it} \, \mathrm{d}t = \int_0^{2\pi} \frac{ire^{it}}{re^{it} - a} \, \mathrm{d}t. \qquad (\dagger)$$

Although this Riemann integral appears innocuous, it turns out to be exceedingly difficult to compute. As we will see next class, there are tricks to avoid calculating it directly. However, it is worth including the direct calculation as a supplement just to prove that this can be done.

Supplement: Direct Calculation of (†)

Observe that if the imaginary unit i were absent (and assuming $a \in \mathbb{R}$) we would find

$$\int_0^{2\pi} \frac{re^t}{re^t - a}\,\mathrm{d}t = \log(re^t - a)\Big|_{t=0}^{t=2\pi}$$

where the log is a natural logarithm. However, the inclusion of the imaginary unit i means that we cannot simply say

$$\int_0^{2\pi} \frac{ire^{it}}{re^{it} - a}\,\mathrm{d}t = \mathrm{Log}(re^{it} - a)\Big|_{t=0}^{t=2\pi} \tag{$*$}$$

where the Log is the principal value of the logarithm. In fact, observe that

$$\mathrm{Log}(re^{it} - a)\Big|_{t=0}^{t=2\pi} = \mathrm{Log}(re^{i2\pi} - a) - \mathrm{Log}(r - a) = \mathrm{Log}(r - a) - \mathrm{Log}(r - a) = 0$$

and so if $(*)$ were actually true, we would find

$$\int_C \frac{1}{z - a}\,\mathrm{d}z = \int_0^{2\pi} \frac{ire^{it}}{re^{it} - a}\,\mathrm{d}t = 0.$$

But we have already shown that in the case $a = 0$, the integral does not equal 0, but rather

$$\int_C \frac{1}{z}\,\mathrm{d}z = 2\pi i.$$

This means that in order to compute

$$\int_0^{2\pi} \frac{ire^{it}}{re^{it} - a}\,\mathrm{d}t$$

we must consider the real and imaginary parts separately. Now,

$$\begin{aligned}
\frac{re^{it}}{re^{it} - a} = \frac{re^{it}}{re^{it} - a} \cdot \frac{re^{-it} - \bar{a}}{re^{-it} - \bar{a}} &= \frac{r^2 - \bar{a}re^{it}}{r^2 - \bar{a}re^{it} - are^{-it} + |a|^2} \\
&= \frac{r^2 - |a|re^{i(t - \mathrm{Arg}\,a)}}{r^2 + |a|^2 - 2|a|r\cos(t - \mathrm{Arg}\,a)} \\
&= \frac{r^2 - |a|r\cos(t - \mathrm{Arg}\,a)}{r^2 + |a|^2 - 2|a|r\cos(t - \mathrm{Arg}\,a)} - i\frac{|a|r\sin(t - \mathrm{Arg}\,a)}{r^2 + |a|^2 - 2|a|r\cos(t - \mathrm{Arg}\,a)}
\end{aligned}$$

so that

$$\begin{aligned}
&\int_0^{2\pi} \frac{ire^{it}}{re^{it} - a}\,\mathrm{d}t \\
&\quad = \int_0^{2\pi} \frac{|a|r\sin(t - \mathrm{Arg}\,a)}{r^2 + |a|^2 - 2|a|r\cos(t - \mathrm{Arg}\,a)}\,\mathrm{d}t + i\int_0^{2\pi} \frac{r^2 - |a|r\cos(t - \mathrm{Arg}\,a)}{r^2 + |a|^2 - 2|a|r\cos(t - \mathrm{Arg}\,a)}\,\mathrm{d}t.
\end{aligned}$$

Now,

$$\int_0^{2\pi} \frac{|a|r\sin(t-\operatorname{Arg}a)}{r^2+|a|^2-2|a|r\cos(t-\operatorname{Arg}a)} = \frac{1}{2}\log(r^2+|a|^2-2|a|r\cos(t-\operatorname{Arg}a))\Big|_{t=0}^{t=2\pi}$$
$$= \frac{1}{2}\log(r^2+|a|^2-2|a|r\cos(2\pi-\operatorname{Arg}a)) - \frac{1}{2}\log(r^2+|a|^2-2|a|r\cos(0-\operatorname{Arg}a))$$
$$= \frac{1}{2}\log(r^2+|a|^2-2|a|r\cos(\operatorname{Arg}a)) - \frac{1}{2}\log(r^2+|a|^2-2|a|r\cos(\operatorname{Arg}a))$$
$$= 0$$

since $r > |a|$. (Recall that a is *inside* C. This is crucial in order for the logarithms to be natural logarithms of positive real numbers.) However, it is rather tricky to compute

$$\int_0^{2\pi} \frac{r^2-|a|r\cos(t-\operatorname{Arg}a)}{r^2+|a|^2-2|a|r\cos(t-\operatorname{Arg}a)}\,\mathrm{d}t = \int_{-\operatorname{Arg}a}^{2\pi-\operatorname{Arg}a} \frac{r^2-|a|r\cos t}{r^2+|a|^2-2|a|r\cos t}\,\mathrm{d}t.$$

One way of doing it is as follows. Observe that

$$\int \frac{-2|a|r\cos t}{r^2+|a|^2-2|a|r\cos t}\,\mathrm{d}t = \int \frac{r^2+|a|^2-2|a|r\cos t}{r^2+|a|^2-2|a|r\cos t}\,\mathrm{d}t - \int \frac{r^2+|a|^2}{r^2+|a|^2-2|a|r\cos t}\,\mathrm{d}t$$
$$= \int 1\,\mathrm{d}t - \int \frac{r^2+|a|^2}{r^2+|a|^2-2|a|r\cos t}\,\mathrm{d}t$$
$$= t - \int \frac{r^2+|a|^2}{r^2+|a|^2-2|a|r\cos t}\,\mathrm{d}t$$

so that

$$\int \frac{-|a|r\cos t}{r^2+|a|^2-2|a|r\cos t}\,\mathrm{d}t = \frac{t}{2} - \frac{1}{2}\int \frac{r^2+|a|^2}{r^2+|a|^2-2|a|r\cos t}\,\mathrm{d}t$$

which in turn implies that

$$\int \frac{r^2-|a|r\cos t}{r^2+|a|^2-2|a|r\cos t}\,\mathrm{d}t = \int \frac{r^2}{r^2+|a|^2-2|a|r\cos t}\,\mathrm{d}t + \frac{t}{2} - \frac{1}{2}\int \frac{r^2+|a|^2}{r^2+|a|^2-2|a|r\cos t}\,\mathrm{d}t$$
$$= \frac{t}{2} + \frac{1}{2}\int \frac{r^2-|a|^2}{r^2+|a|^2-2|a|r\cos t}\,\mathrm{d}t.$$

Recall that $\cos t = \cos^2(t/2) - \sin^2(t/2)$ and $1 = \cos^2(t/2) + \sin^2(t/2)$ so that

$$\frac{1}{2}\int \frac{r^2-|a|^2}{r^2+|a|^2-2|a|r\cos t}\,\mathrm{d}t$$
$$= \frac{1}{2}\int \frac{r^2-|a|^2}{r^2+|a|^2-2|a|r(\cos^2(t/2)-\sin^2(t/2))}\,\mathrm{d}t$$
$$= \frac{1}{2}\int \frac{r^2-|a|^2}{(r^2+|a|^2)(\cos^2(t/2)+\sin^2(t/2))-2|a|r(\cos^2(t/2)-\sin^2(t/2))}\,\mathrm{d}t$$
$$= \frac{1}{2}\int \frac{r^2-|a|^2}{(r^2+|a|^2-2|a|r)\cos^2(t/2)+(r^2+|a|^2+2|a|r)\sin^2(t/2)}\,\mathrm{d}t$$
$$= \frac{1}{2}\int \frac{r^2-|a|^2}{(r-|a|)^2\cos^2(t/2)+(r+|a|)^2\sin^2(t/2)}\,\mathrm{d}t$$

which upon further simplification yields

$$\begin{aligned}\frac{1}{2}\int \frac{r^2-|a|^2}{r^2+|a|^2-2|a|r\cos t}\,\mathrm{d}t &= \frac{r^2-|a|^2}{2(r-|a|)^2}\int \frac{\sec^2(t/2)}{1+\left(\frac{r+|a|}{r-|a|}\tan(t/2)\right)^2}\,\mathrm{d}t \\ &= \frac{r+|a|}{2(r-|a|)}\int \frac{\sec^2(t/2)}{1+\left(\frac{r+|a|}{r-|a|}\tan(t/2)\right)^2}\,\mathrm{d}t.\end{aligned}$$

Make the substitution

$$\theta = \frac{r+|a|}{r-|a|}\tan(t/2) \quad \text{so that} \quad \mathrm{d}\theta = \frac{r+|a|}{2(r-|a|)}\sec^2(t/2)\,\mathrm{d}t$$

which implies

$$\frac{r+|a|}{2(r-|a|)}\int \frac{\sec^2(t/2)}{1+\left(\frac{r+|a|}{r-|a|}\tan(t/2)\right)^2}\,\mathrm{d}t = \int \frac{1}{1+\theta^2}\,\mathrm{d}\theta = \arctan\theta = \arctan\left(\frac{r+|a|}{r-|a|}\tan(t/2)\right).$$

Hence,

$$\int \frac{r^2-|a|r\cos t}{r^2+|a|^2-2|a|r\cos t}\,\mathrm{d}t = \frac{t}{2} + \arctan\left(\frac{r+|a|}{r-|a|}\tan(t/2)\right).$$

However, if we want to compute the definite integral

$$\int_0^{2\pi} \frac{r^2-|a|r\cos(t-\operatorname{Arg} a)}{r^2+|a|^2-2|a|r\cos(t-\operatorname{Arg} a)}\,\mathrm{d}t = \int_{-\operatorname{Arg} a}^{2\pi-\operatorname{Arg} a} \frac{r^2-|a|r\cos t}{r^2+|a|^2-2|a|r\cos t}\,\mathrm{d}t$$

we cannot just write

$$\int_{-\operatorname{Arg} a}^{2\pi-\operatorname{Arg} a} \frac{r^2-|a|r\cos t}{r^2+|a|^2-2|a|r\cos t}\,\mathrm{d}t = \left[\frac{t}{2} + \arctan\left(\frac{r+|a|}{r-|a|}\tan(t/2)\right)\right]_{t=-\operatorname{Arg} a}^{t=2\pi-\operatorname{Arg} a}.$$

The reason for this is that the definite integral on the left in the previous expression is actually improper. This can be seen by considering the expression on the right. The trouble spot is when $t=\pi$; that is, $\tan(\pi/2)$ is not defined, and so we cannot just compute the integral over the range $-\operatorname{Arg} a \le t \le 2\pi - \operatorname{Arg} a$ without considering what happens when $t=\pi$. Since $\operatorname{Arg} a \in (-\pi,\pi]$, we can conclude that $-\operatorname{Arg} a \le \pi \le 2\pi - \operatorname{Arg} a$ so that the trouble spot is actually in the range of integration. That is,

$$\begin{aligned}&\int_{-\operatorname{Arg} a}^{2\pi-\operatorname{Arg} a} \frac{r^2-|a|r\cos t}{r^2+|a|^2-2|a|r\cos t}\,\mathrm{d}t \\ &\qquad= \int_{-\operatorname{Arg} a}^{\pi} \frac{r^2-|a|r\cos t}{r^2+|a|^2-2|a|r\cos t}\,\mathrm{d}t + \int_{\pi}^{2\pi-\operatorname{Arg} a} \frac{r^2-|a|r\cos t}{r^2+|a|^2-2|a|r\cos t}\,\mathrm{d}t \\ &\qquad= \lim_{\theta\uparrow\pi}\int_{-\operatorname{Arg} a}^{\theta} \frac{r^2-|a|r\cos t}{r^2+|a|^2-2|a|r\cos t}\,\mathrm{d}t + \lim_{\theta\downarrow\pi}\int_{\theta}^{2\pi-\operatorname{Arg} a} \frac{r^2-|a|r\cos t}{r^2+|a|^2-2|a|r\cos t}\,\mathrm{d}t.\end{aligned}$$

Now

$$\lim_{\theta\uparrow\pi}\int_{-\operatorname{Arg} a}^{\theta}\frac{r^2-|a|r\cos t}{r^2+|a|^2-2|a|r\cos t}\,\mathrm{d}t$$

$$=\lim_{\theta\uparrow\pi}\left[\frac{t}{2}+\arctan\left(\frac{r+|a|}{r-|a|}\tan(t/2)\right)\right]_{t=-\operatorname{Arg} a}^{t=\theta}$$

$$=\frac{\pi}{2}+\frac{\operatorname{Arg} a}{2}-\arctan\left(\frac{r+|a|}{r-|a|}\tan\left(-\frac{\operatorname{Arg} a}{2}\right)\right)+\lim_{\theta\uparrow\pi}\arctan\left(\frac{r+|a|}{r-|a|}\tan(t/2)\right)\qquad(*)$$

and

$$\lim_{\theta\downarrow\pi}\int_{\theta}^{2\pi-\operatorname{Arg} a}\frac{r^2-|a|r\cos t}{r^2+|a|^2-2|a|r\cos t}\,\mathrm{d}t$$

$$=\lim_{\theta\downarrow\pi}\left[\frac{t}{2}+\arctan\left(\frac{r+|a|}{r-|a|}\tan(t/2)\right)\right]_{t=\theta}^{t=2\pi-\operatorname{Arg} a}$$

$$=\frac{\pi}{2}-\frac{\operatorname{Arg} a}{2}+\arctan\left(\frac{r+|a|}{r-|a|}\tan\left(\pi-\frac{\operatorname{Arg} a}{2}\right)\right)-\lim_{\theta\downarrow\pi}\arctan\left(\frac{r+|a|}{r-|a|}\tan(t/2)\right)\qquad(**)$$

so that adding $(*)$ and $(**)$ and using the fact that $\tan(-\theta)=\tan(\pi-\theta)$ implies

$$\int_{-\operatorname{Arg} a}^{2\pi-\operatorname{Arg} a}\frac{r^2-|a|r\cos t}{r^2+|a|^2-2|a|r\cos t}\,\mathrm{d}t$$

$$=\pi+\lim_{\theta\uparrow\pi}\arctan\left(\frac{r+|a|}{r-|a|}\tan(t/2)\right)-\lim_{\theta\downarrow\pi}\arctan\left(\frac{r+|a|}{r-|a|}\tan(t/2)\right)$$

Since $\tan(t/2)\to\infty$ as $t\uparrow\pi$ and since $\tan(t/2)\to-\infty$ as $t\downarrow\pi$, we conclude

$$\lim_{\theta\uparrow\pi}\arctan\left(\frac{r+|a|}{r-|a|}\tan(t/2)\right)=\frac{\pi}{2}$$

and

$$\lim_{\theta\downarrow\pi}\arctan\left(\frac{r+|a|}{r-|a|}\tan(t/2)\right)=-\frac{\pi}{2}$$

so that

$$\int_{-\operatorname{Arg} a}^{2\pi-\operatorname{Arg} a}\frac{r^2-|a|r\cos t}{r^2+|a|^2-2|a|r\cos t}\,\mathrm{d}t=\pi+\frac{\pi}{2}+\frac{\pi}{2}=2\pi.$$

In summary, we have shown that

$$\int_C\frac{1}{z-a}\,\mathrm{d}z=\int_0^{2\pi}\frac{ire^{it}}{re^{it}-a}\,\mathrm{d}t=2\pi i$$

if a is inside $C=\{|z|=r\}$, the circle of radius $r>0$ centred at 0 oriented counterclockwise.