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Lecture #21: The Cauchy Integral Theorem

Recall from last lecture that we proved the following theorem.

Theorem 21.1 (Fundamental Theorem of Calculus for Integrals over Closed Contours)**.** *Suppose that D is a domain. If $f(z)$ is continuous in D and has an antiderivative $F(z)$ throughout D (i.e., $F(z)$ is analytic in D with $F'(z) = f(z)$ for every $z \in D$), then*

$$\int_C f(z)\,\mathrm{d}z = 0$$

for any closed contour C lying entirely in D.

Remark. This theorem can apply if D is an annulus and C surrounds the hole.

Theorem 21.2 (Cauchy Integral Theorem, Basic Version)**.** *Suppose that D is a domain. If $f(z)$ is analytic in D and $f'(z)$ is continuous throughout D, then*

$$\int_C f(z)\,\mathrm{d}z = 0$$

for any closed contour C lying entirely in D having the property that the region surrounded by C is a simply connected subdomain of D (in other words, C is continuously deformable to a point.)

This follows from Green's theorem and requires the assumptions that $f'(z)$ be continuous throughout D and C be continuously deformable to a point. Recall that Green's theorem is usually stated as follows.

Theorem 21.3 (Green's Theorem)**.** *Suppose that R is a simply connected domain and that $C = \partial R$ is a closed contour oriented counterclockwise. Let $P = P(x,y) : R \to \mathbb{R}$, $Q = Q(x,y) : R \to \mathbb{R}$ be continuously differentiable in R (so that P_x, P_y, Q_x, Q_y are continuous in R). Then,*

$$\int_C P(x,y)\,\mathrm{d}x + Q(x,y)\,\mathrm{d}y = \iint_R \left(\frac{\partial Q(x,y)}{\partial x} - \frac{\partial P(x,y)}{\partial y} \right) \mathrm{d}x\,\mathrm{d}y.$$

Proof of Cauchy Integral Theorem, Basic Version. Suppose that D is a domain and C is a closed contour in D which is continuously deformable to a point so that R, the interior of C, is a simply connected subdomain of D. Suppose further that $f(z)$ is analytic in D and that $f'(z)$ is continuous in D. In particular, this means that $f(z)$ is analytic in R, and that $f'(z)$ is continuous in R. Thus, if we write $f(z) = u(x,y) + iv(x,y)$ for $z \in D$ and $\mathrm{d}z = \mathrm{d}x + i\,\mathrm{d}y$, then

$$\begin{aligned}
\int_C f(z)\,\mathrm{d}z &= \int_C (u(x,y) + iv(x,y))(\mathrm{d}x + i\,\mathrm{d}y) \\
&= \int_C u(x,y)\,\mathrm{d}x + iv(x,y)\,\mathrm{d}x + iu(x,y)\,\mathrm{d}y - v(x,y)\,\mathrm{d}y \\
&= \int_C u(x,y)\,\mathrm{d}x - v(x,y)\,\mathrm{d}y + i\int_C v(x,y)\,\mathrm{d}x + u(x,y)\,\mathrm{d}y.
\end{aligned}$$

Without loss of generality, assume that C is oriented counterclockwise. Since $C = \partial R$ is a closed contour, R is a simply connected domain, and u_x, u_y, v_x, v_y are continuous (since $f(z)$ is analytic in R and $f'(z)$ is continuous in R), we can apply Green's theorem to each integral separately. That is,

$$\begin{aligned}\int_C u(x,y)\,\mathrm{d}x - v(x,y)\,\mathrm{d}y &= \iint_R \left(-\frac{\partial v(x,y)}{\partial x} - \frac{\partial u(x,y)}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y \\ &= -\iint_R \left(\frac{\partial v(x,y)}{\partial x} + \frac{\partial u(x,y)}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y \qquad (*)\end{aligned}$$

and

$$\int_C v(x,y)\,\mathrm{d}x + u(x,y)\,\mathrm{d}y = \iint_R \left(\frac{\partial u(x,y)}{\partial x} - \frac{\partial v(x,y)}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y. \qquad (**)$$

However, since $f(z)$ is analytic in D we know that the Cauchy-Riemann equations are satisfied in D; that is,

$$u_x(x_0,y_0) = v_y(x_0,y_0) \quad \text{and} \quad u_y(x_0,y_0) = -v_x(x_0,y_0)$$

for any $z_0 = x_0 + iy_0 \in D$. This implies that

$$\iint_R \left(\frac{\partial v(x,y)}{\partial x} + \frac{\partial u(x,y)}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y = 0 \quad \text{and} \quad \iint_R \left(\frac{\partial u(x,y)}{\partial x} - \frac{\partial v(x,y)}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y = 0$$

so that $(*)$ and $(**)$ imply

$$\int_C f(z)\,\mathrm{d}z = \int_C u(x,y)\,\mathrm{d}x - v(x,y)\,\mathrm{d}y + i\int_C v(x,y)\,\mathrm{d}x + u(x,y)\,\mathrm{d}y = 0$$

as required. □

Remark. This theorem cannot apply if D is an annulus and C surrounds the hole.

Example 21.4. Suppose that $D = \{1 < |z| < 3\}$ is the interior of the annulus of inner radius 1 and outer radius 3. Let $C = \{|z| = 2\}$ denote the circle of radius 2. Show

$$\int_C 3z^2\,\mathrm{d}z = 0.$$

Solution. Observe that D is a domain. Also observe that $f(z) = 3z^2$ is continuous in D and has antiderivative $F(z) = z^3$ throughout D; that is, $F(z) = z^3$ is analytic in D with $F'(z) = 3z^2 = f(z)$. The contour C is closed and lies entirely in D. Thus, the hypotheses have been met for the Fundamental Theorem of Calculus for Integrals over Closed Contours and so we conclude

$$\int_C 3z^2\,\mathrm{d}z = 0.$$

Note that we **cannot** apply the Cauchy Integral Theorem to solve this problem. It is true that $f(z) = 3z^2$ is analytic in D with $f'(z) = 6z$ so that $f'(z)$ is continuous in D. It is also true that the contour C lies entirely in D. However, C is *not* continuously deformable to a point; in other words, the interior of C is *not* a simply connected subdomain of D. In fact, if $R \subset D$ denotes the interior of C, then $R = \{1 < |z| < 2\}$. Thus, the hypotheses for the Cauchy Integral Theorem have not been met.

Example 21.5. Suppose that $D = \{|z| < 3\}$ is the interior of the disk of radius 3. Let $C = \{|z| = 2\}$ denote the circle of radius 2. Show

$$\int_C 3z^2 \, \mathrm{d}z = 0.$$

Solution. In this case, since D is a domain and the closed contour C is continuously deformable to a point, we can apply the Cauchy Integral Theorem. That is, $f(z) = 3z^2$ is analytic in D and the interior of C is $\{|z| < 2\}$ which is a simply connected subdomain of D. Therefore, by the Cauchy Integral Theorem, Basic Version, we conclude

$$\int_C 3z^2 \, \mathrm{d}z = 0.$$

Of course, we could also use the Fundamental Theorem of Calculus for Integrals over Closed Contours to draw the same conclusion. That is, $f(z) = 3z^2$ is continuous in D and has antiderivative $F(z) = z^3$ throughout D, so that

$$\int_C 3z^2 \, \mathrm{d}z = 0$$

since the hypotheses of Theorem 20.3 have been met.

Remark. The Cauchy Integral Theorem in the form we stated it was first proved by Augustin-Louis Cauchy (1789–1857). It was later shown by Édouard Goursat (1858–1936) that the assumption that $f'(z)$ be continuous is unnecessary.

Theorem 21.6 (Cauchy Integral Theorem, Advanced Version)**.** *Suppose that D is a domain. If $f(z)$ is analytic in D, then*

$$\int_C f(z) \, \mathrm{d}z = 0$$

for any closed contour C lying entirely in D having the property that the region surrounded by C is a simply connected subset of D.

Remark. The proof of this theorem is much too sophisticated for Math 312. However, for the purposes of this class, any time you are asked to use the Cauchy Integral Theorem, you will be able to verify that $f'(z)$ is continuous.

Corollary 21.7 (Cauchy Integral Theorem for Simply Connected Domains, Basic Version)**.** *Suppose that D is a simply connected domain. If $f(z)$ is analytic in D, $f'(z)$ is continuous in D, and C is a closed contour lying entirely in D, then*

$$\int_C f(z) \, \mathrm{d}z = 0.$$

Corollary 21.8 (Cauchy Integral Theorem for Simply Connected Domains, Advanced Version)**.** *Suppose that D is a simply connected domain. If $f(z)$ is analytic in D and C is a closed contour lying entirely in D, then*

$$\int_C f(z) \, \mathrm{d}z = 0.$$

Proof of both corollaries. If D is simply connected, then any closed contour lying entirely in D is necessarily continuously deformable to a point. □