Lecture #20: Analyticity of the Complex Logarithm Function

Definition. Suppose that $z \in \mathbb{C} \setminus \{0\}$. We define the *principal value of the logarithm of* z, denoted $\operatorname{Log} z$, to be

$$\operatorname{Log} z = \log|z| + i\operatorname{Arg}(z).$$

Proposition 20.1. *The function $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ given by $f(z) = \operatorname{Log} z$ is continuous at all z except those along the negative real axis.*

Proof. Since $z \mapsto \log|z|$ is clearly continuous for all $z \in \mathbb{C} \setminus \{0\}$ and since $\operatorname{Log} z = \log|z| + i\operatorname{Arg}(z)$, the result follows from the fact that $z \mapsto \operatorname{Arg}(z)$ is discontinuous at each point on the nonpositive real axis. That is, let $z = x_0 + iy$ for some $x_0 < 0$ fixed. If $y \downarrow 0$, then $\operatorname{Arg}(z) \downarrow \pi$, whereas if $y \uparrow 0$, then $\operatorname{Arg}(z) \uparrow -\pi$. □

Recall that if $f : (0, \infty) \to \mathbb{R}$ is given by $f(x) = \log x$, then $f'(x) = 1/x$. The same type of formula holds for the principal value of the logarithm, but must be stated very carefully.

Theorem 20.2. *The function $z \mapsto \operatorname{Log} z$ is analytic in the domain $D = \mathbb{C} \setminus D^*$ where*

$$D^* = \{z \in \mathbb{C} : \operatorname{Re}(z) \le 0 \;\; \textit{and} \;\; \operatorname{Im}(z) = 0\}$$

and satisfies

$$\frac{\mathrm{d}}{\mathrm{d}z} \operatorname{Log} z = \frac{1}{z}$$

for $z \in D$.

Proof. Let $w = \operatorname{Log} z$. We must show that

$$\lim_{z \to z_0} \frac{w - w_0}{z - z_0}$$

exists and equals $1/z_0$ for every $z_0 \in D$. However, we know (by definition of $\operatorname{Log} z$) that $z = e^w$. We also know from Example 15.2 that $f(w) = e^w$ is entire with $f'(w) = e^w$. In other words,

$$\frac{\mathrm{d}}{\mathrm{d}w} f(w)\bigg|_{w=w_0} = \frac{\mathrm{d}}{\mathrm{d}w} e^w\bigg|_{w=w_0} = \frac{\mathrm{d}z}{\mathrm{d}w}\bigg|_{w=w_0} = \lim_{w \to w_0} \frac{z - z_0}{w - w_0} = e^{w_0} = z_0. \qquad (*)$$

The next step is to observe that by continuity (Proposition 20.1), $w \to w_0$ as $z \to z_0$. Hence,

$$\lim_{z \to z_0} \frac{w - w_0}{z - z_0} = \lim_{w \to w_0} \frac{w - w_0}{z - z_0}. \qquad (**)$$

However, compare the right side of $(**)$ with $(*)$ to conclude

$$\frac{\mathrm{d}}{\mathrm{d}z} \operatorname{Log} z\bigg|_{z=z_0} = \lim_{z \to z_0} \frac{w - w_0}{z - z_0} = \lim_{w \to w_0} \frac{w - w_0}{z - z_0} = \lim_{w \to w_0} \frac{1}{\frac{z - z_0}{w - w_0}} = \frac{1}{z_0}$$

for every $z_0 \in D$. □

Remark. Assuming appropriate smoothness, we have shown that the real part of *every* analytic function f is harmonic. The converse, however, is not true. That is, not every smooth harmonic function $u : D \to \mathbb{R}$ is necessarily the real part of some analytic function. As an example, consider $u(z) = \log|z|$ for $z \in D = \{0 < |z| < 1\}$. It is not hard to show that u is harmonic in D. However, it can also be shown that u does *not* have a harmonic conjugate in D. Compare this to Problem #10 on Assignment #4. The function $u(z) = \log|z|$ for $z \in D = \{\mathrm{Re}\, z > 0\}$ is harmonic in D and does have a harmonic conjugate in D.

The Cauchy Integral Theorem

Our next goal is to investigate the conditions under which

$$\int_C f(z)\,\mathrm{d}z = 0$$

for a closed contour C.

Theorem 20.3 (Fundamental Theorem of Calculus for Integrals over Closed Contours)**.** *Suppose that D is a domain. If $f(z)$ is continuous in D and has an antiderivative $F(z)$ throughout D (i.e., $F(z)$ is analytic in D with $F'(z) = f(z)$ for every $z \in D$), then*

$$\int_C f(z)\,\mathrm{d}z = 0$$

for any closed contour C lying entirely in D.

Proof. This follows from the usual Fundamental Theorem of Calculus. Suppose that C is parametrized by $z = z(t)$, $a \le t \le b$. The hypothesis that $f(z)$ is continuous in D is necessary for the contour integral

$$\int_C f(z)\,\mathrm{d}z$$

to equal the Riemann integral

$$\int_a^b f(z(t)) \cdot z'(t)\,\mathrm{d}t.$$

The assumption that f has an antiderivative F means that

$$\frac{\mathrm{d}}{\mathrm{d}t} F(z(t)) = F'(z(t)) \cdot z'(t) = f(z(t)) \cdot z'(t).$$

Therefore,

$$\int_C f(z)\,\mathrm{d}z = \int_a^b f(z(t)) \cdot z'(t)\,\mathrm{d}t = \int_a^b \frac{\mathrm{d}}{\mathrm{d}t} F(z(t))\,\mathrm{d}t = F(z(b)) - F(z(a))$$

by the usual Fundamental Theorem of Calculus. The assumption that C is a closed contour means that $z(a) = z(b)$ which implies $F(z(b)) = F(z(a))$. Hence,

$$\int_C f(z)\,\mathrm{d}z = 0$$

for any closed contour C lying entirely in D. □

Remark. This theorem can apply if D is an annulus and C surrounds the hole.