Lecture #1: Introduction to Complex Variables

In calculus, we study

- algebraic operations with real numbers,
- functions, limits, continuity, graphing,
- differentiation and applications,
- integration and applications, and
- series and sequences.

In complex analysis, we will develop these topics in a parallel manner. We will call $z = a+ib$, with $i = \sqrt{-1}$ and $a, b \in \mathbb{R}$, a complex variable and will study

- properties of the complex plane and algebraic operations with complex variables,
- properties of functions $f(z)$, limits, graphing, differentiation, and
- integration of function $f(z)$, say $\int_C f(z)\,\mathrm{d}z$ where C is some curve in the complex plane.

Algebra of Complex Variables

The motivation for introducing $i = \sqrt{-1}$ is to *solve* the equation $x^2 + 1 = 0$. In general, the fact that quadratic equations can have no real roots motivates introducing complex variables.

Notation. A *complex variable* z is of the form $z = a + ib$ where a and b are real numbers.

Definition. Two complex variables $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ are *equal* if and only if $a_1 = a_2$ and $b_1 = b_2$.

Notation. Let $z = a + ib$ be a complex variable. The real part of z is $\mathrm{Re}(z) = a$ and the imaginary part of z is $\mathrm{Im}(z) = b$.

Fact. The complex variables z_1 and z_2 are equal iff $\mathrm{Re}(z_1) = \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) = \mathrm{Im}(z_2)$.

Note that we are using the phrase *complex variable* instead of *complex number*. This is because we wish to stress that $z = a+ib$ is *not* a number in the usual, or real, sense. Instead it is an object that we have created. The power of complex variables will become apparent when we apply complex methods to solve real problems. Indeed,

> "...the shortest and best way between two truths of the real domain often passes through the imaginary one."
> —Jacques Hadamard (1865–1963)

Arithmetic of Complex Variables

Let $i = \sqrt{-1}$, $z_1 = a_1 + ib_1$, and $z_2 = a_2 + ib_2$ with $a_1, a_2, b_1, b_2 \in \mathbb{R}$. We define the operations of addition, multiplication, and division (provided either $a_2 \neq 0$ or $b_2 \neq 0$) as follows.

Addition. $z_1 + z_2 = (a_1 + a_2) + i(b_1 + b_2)$

Multiplication. $z_1 z_2 = (a_1 + ib_1)(a_2 + ib_2) = (a_1 a_2 - b_1 b_2) + i(b_1 a_2 + a_1 b_2)$

Division. $\dfrac{z_1}{z_2} = \dfrac{a_1 + ib_1}{a_2 + ib_2} = \dfrac{a_1 + ib_1}{a_2 + ib_2}\dfrac{a_2 - ib_2}{a_2 - ib_2} = \dfrac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2} + i\dfrac{b_1 a_2 - a_1 b_2}{a_2^2 + b_2^2}$

Remark. One way to remember these definitions is to manipulate the expressions just as you would for real numbers, but replacing i^2 by -1. For example,

$$(a_1 + ib_1)(a_2 + ib_2) = a_1 a_2 + i a_1 b_2 + i b_1 a_2 + i^2 b_1 b_2 = a_1 a_2 - b_1 b_2 + i(b_1 a_2 + a_1 b_2).$$

The key is that the motivation for making the definitions we have comes from our experience with real numbers. However, there is no underlying reason why these expressions for addition, multiplication, and division of complex variables are true. They are simply definitions.

It can now be easily shown that if addition, multiplication, and division are defined in this way, then the following hold for complex variables z_1, z_2, z_3.

Commutative Law. $z_1 + z_2 = z_2 + z_1$ and $z_1 z_2 = z_2 z_1$

Associative Law. $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$ and $(z_1 z_2) z_3 = z_1 (z_2 z_3)$

Distributive Law. $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$

Exercise 1.1. Verify the commutative law, associative law, and distributive law hold for complex variables.

Remark. Consider the complex variable $z = a + ib$. We have $z = 0$ iff $a = 0$ and $b = 0$. Note that the complex variable 0 is shorthand for the complex variable $0 + i0$. Moreover, the real number a can be identified with the complex variable $a + i0$. We will, however, write this complex variable simply as a.

Proposition 1.2. *Consider the complex variables $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$. If $z_1 z_2 = 0$, then either $z_1 = 0$ or $z_2 = 0$.*

Proof. Since $z_1 z_2 = (a_1 a_2 - b_1 b_2) + i(b_1 a_2 + a_1 b_2) = 0$ we conclude that

$$a_1 a_2 - b_1 b_2 = 0 \quad \text{and} \quad b_1 a_2 + a_1 b_2 = 0.$$

An equivalent way to write this system of equations is in matrix notation as

$$\begin{bmatrix} a_2 & -b_2 \\ b_2 & a_2 \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

In order to complete the proof, we will show that if $z_2 \neq 0$, then z_1 must be 0. Therefore, assume that $z_2 \neq 0$ so that either $a_2 \neq 0$ or $b_2 \neq 0$. In particular, this implies that

$$\det \begin{bmatrix} a_2 & -b_2 \\ b_2 & a_2 \end{bmatrix} = a_2^2 + b_2^2 > 0.$$

However, recall from Math 122 that the only solution to the matrix system $A\mathbf{v} = \mathbf{0}$ with $\det A > 0$ is $\mathbf{v} = \mathbf{0}$. This implies that $a_1 = b_1 = 0$ so $z_1 = 0$ as required. □

We end this discussion with one more convention concerning complex variables that is motivated by the arithmetic of real numbers. If k is a positive integer and z is a complex variable, then the power or exponential z^k is shorthand for multiplication of z by itself k times; for instance,

$$z^4 = zzzz.$$

We can then compute the product $zzzz$ using the associative law and the definition of multiplication of complex variables along with the identities $i^2 = -1$, $i^3 = -i$, and $i^4 = 1$. If m is a negative integer, say $m = -k$ for some non-negative integer k, define $z^m = z^{-k}$ by

$$z^{-k} = \frac{1}{z^k}.$$

Finally, take $z^0 = 1$.

Exercise 1.3. Verify that if $z = 1 + i$, then $z^4 = -4$.