Lecture #19: Contour Integration

Example 19.1. Compute

$$I_1 = \int_{C_1} \overline{z} \, \mathrm{d}z$$

if $C_1 = \{e^{it}, 0 \le t \le \pi\}$ is that part of the upper half of the unit circle going from 1 to -1.

Solution. If $z(t) = e^{it}$, $0 \le t \le \pi$, then $z'(t) = ie^{it}$, and so

$$\int_{C_1} \overline{z} \, \mathrm{d}z = \int_0^\pi \overline{z(t)} \cdot z'(t) \, \mathrm{d}t = \int_0^\pi e^{-it} \cdot ie^{it} \, \mathrm{d}t = i \int_0^\pi \mathrm{d}t = i\pi.$$

Example 19.2. Compute

$$I_2 = \int_{C_2} \overline{z} \, \mathrm{d}z$$

if $C_2 = \{e^{-it}, 0 \le t \le \pi\}$ is that part of the lower half of the unit circle going from 1 to -1.

Solution. If $z(t) = e^{-it}$, $0 \le t \le \pi$, then $z'(t) = -ie^{it}$, and so

$$\int_{C_2} \overline{z} \, \mathrm{d}z = \int_0^\pi \overline{z(t)} \cdot z'(t) \, \mathrm{d}t = \int_0^\pi e^{it} \cdot -ie^{-it} \, \mathrm{d}t = -i \int_0^\pi \mathrm{d}t = -i\pi.$$

Note that the answers to the previous two examples are different; that is, even though the contours C_1 and C_2 start and end at the same points, $I_1 \neq I_2$. What is the difference between this pair of examples and the pair of examples from last lecture?

Theorem 19.3 (Fundamental Theorem of Calculus for Contour Integrals)**.** *Suppose that D is a domain. If $f(z)$ is continuous in D and has an antiderivative $F(z)$ throughout D (i.e., $F(z)$ is analytic in D with $F'(z) = f(z)$ for every $z \in D$), then*

$$\int_C f(z) \, \mathrm{d}z = F(z(b)) - F(z(a))$$

for any contour C lying entirely in D.

Proof. Suppose that C lies entirely in D and is parametrized by $z = z(t)$, $a \le t \le b$. From the definition of contour integral, we have

$$\int_C f(z) \, \mathrm{d}z = \int_a^b f(z(t)) \cdot z'(t) \, \mathrm{d}t$$

and note that the assumption that $f(z)$ is continuous means that $f(z(t)) \cdot z'(t)$ is Riemann integrable on $[a, b]$. The assumption that f has an antiderivative F means that

$$\frac{\mathrm{d}}{\mathrm{d}t} F(z(t)) = F'(z(t)) \cdot z'(t) = f(z(t)) \cdot z'(t).$$

Therefore,

$$\int_C f(z)\,\mathrm{d}z = \int_a^b f(z(t))\cdot z'(t)\,\mathrm{d}t = \int_a^b \frac{\mathrm{d}}{\mathrm{d}t}F(z(t))\,\mathrm{d}t = F(z(b)) - F(z(a))$$

by the usual Fundamental Theorem of Calculus. □

Example 19.4. Compute

$$\int_C z^2\,\mathrm{d}z$$

where C is any contour connecting 1 and $2+i$.

Solution. Observe that $f(z) = z^2$ is continuous in $\mathbb{C}$ and $F(z) = z^3/3$ is entire with $F'(z) = f(z)$. Therefore, if C is any contour with $z(a) = 1$ and $z(b) = 2+i$, then the Fundamental Theorem of Calculus for Contour Integrals implies

$$\int_C z^2\,\mathrm{d}z = \left.\frac{z^3}{3}\right|_{z=2+i} - \left.\frac{z^3}{3}\right|_{z=1} = \frac{(2+i)^3}{3} - \frac{1}{3} = \frac{1}{3} + \frac{11}{3}i.$$

Remark. This explains why the answers to Examples 18.4 and 18.5 are the same. Note that the function from Examples 19.1 and 19.2, namely $\bar{z}$, does not have an antiderivative. This is why the Fundamental Theorem of Calculus for Contour Integrals does not apply, and so we are not surprised that contour integrals of $\bar{z}$ *do* depend on the contour taken.

Example 19.5. Compute

$$\int_C e^{iz}\,\mathrm{d}z$$

where C is that part of the unit circle in the first quadrant going from 1 to i.

Solution. Observe that $f(z) = e^{iz}$ is continuous in $\mathbb{C}$ and $F(z) = -ie^{iz}$ is entire with $F'(z) = f(z)$. Therefore, since C is a contour with $z(a) = 1$ and $z(b) = i$, the Fundamental Theorem of Calculus for Contour Integrals implies

$$\int_C e^{iz}\,\mathrm{d}z = -ie^{iz}\Big|_{z=i} + ie^{iz}\Big|_{z=1} = -ie^{-1} + ie^{i} = ie^{i} - ie^{-1}.$$

The Complex Logarithm

Recall that we introduced the complex-valued logarithm function in Lecture #15. We will now re-visit that function. For real variables, we can define the *(natural) logarithm* of $x > 0$, written as $\log x$, to be that unique number satisfying $e^{\log x} = x$. Moreover, we also know that $\log(e^x) = x$ so that the functions $f(x) = e^x$ and $g(x) = \log x$ are inverses.

Example 19.6. Solve $e^x = \pi/4$ for $x \in \mathbb{R}$.

Solution. We can use logarithms to solve this problem. That is, $e^x = \pi/4$ implies $x = \log(e^x) = \log(\pi/4)$.

Remark. To solve the previous problem we used a key fact about real-valued logarithms, namely

$$e^{x_1} = e^{x_2} \quad \text{if and only if} \quad x_1 = x_2,$$

or, equivalently,

$$\log x_1 = \log x_2 \quad \text{if and only if} \quad x_1 = x_2.$$

We have already discovered that the function e^z is $2\pi i$ periodic, namely $e^z = e^{z+2\pi i}$, so that we cannot simply define the complex-valued logarithm to be the inverse of e^z.

Example 19.7. Solve $e^z = 1+i$ for $z \in \mathbb{C}$.

Solution. We write $1+i$ in polar coordinates as $1+i = \sqrt{2}e^{i\pi/4}$ so that we need to solve

$$e^z = \sqrt{2}e^{i\pi/4}$$

for z. Consider $e^\zeta = e^{i\pi/4}$. One solution is $\zeta = i\pi/4$. But this is not the only solution. By periodicity, we know $e^\zeta = e^{\zeta+2\pi k i}$ for any $k \in \mathbb{Z}$. Hence,

$$e^{\zeta+2\pi k i} = e^{i\pi/4}$$

implies $\zeta \in \{(\pi/4 + 2\pi k)i, \quad k \in \mathbb{Z}\}$ and so

$$z \in \{\frac{1}{2}\log 2 + (\pi/4 + 2\pi k)i, \quad k \in \mathbb{Z}\}.$$

Let $w \in \mathbb{C} \neq 0$. We know that there are infinitely many values of $z \in \mathbb{C}$ such that $e^z = w$; see Figure 19.1.

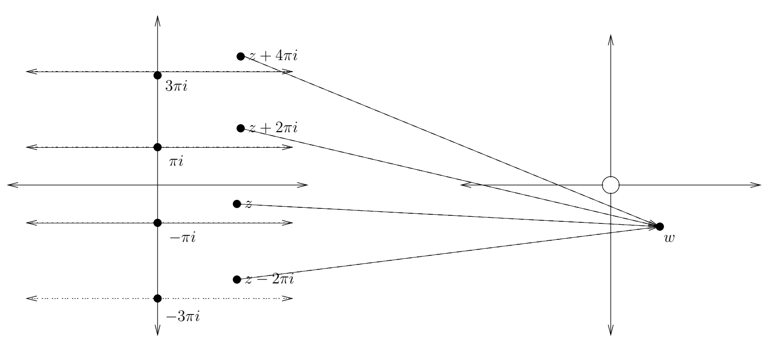

Figure 19.1: The image of $\mathbb{C}$ under the mapping e^z.

However, there is a *unique* value of z in the fundamental region $\{-\pi < \operatorname{Im} z \le \pi\}$ with $e^z = w$. This is what we will use to define the logarithm of w; more precisely, this will be the *principal value of the logarithm.*

Definition. Suppose that $w \in \mathbb{C} \setminus \{0\}$. We define the *principal value of the logarithm of* w, denoted $\operatorname{Log} w$, to be

$$\operatorname{Log} w = \log |w| + i \operatorname{Arg}(w).$$

Remark. We are writing Log with a capital L to stress that it is the principal value of the complex-valued logarithm. Note that $\log x$ for $x \in \mathbb{R}$ denotes the usual real-valued natural logarithm.

Remark. The principal value of the logarithm of $w \neq 0$ can also be defined as the unique value of z with $-\pi < \operatorname{Im} z \leq \pi$ such that $e^z = w$.

Example 19.8. Compute $\operatorname{Log}(1+i)$.

Solution. Since $|1+i| = \sqrt{2}$ and $\operatorname{Arg}(1+i) = \pi/4$, we find

$$\operatorname{Log}(1+i) = \log\sqrt{2} + i\pi/4 = \frac{1}{2}\log 2 + i\frac{\pi}{4}.$$

Definition. Let $w \in \mathbb{C} \setminus \{0\}$. The *complex-valued logarithm of* w is the multiple-valued function given by

$$\log w = \log|w| + i\arg(w).$$

Note that this is an equality of sets; since $\arg(w) = \{\operatorname{Arg}(w) + 2\pi k, k \in \mathbb{Z}\}$, we can also write

$$\log w = \{\log|w| + i\operatorname{Arg}(w) + 2\pi k i, k \in \mathbb{Z}\}.$$

Recall from Assignment #1 that $\arg(w_1 w_2) = \arg(w_1) + \arg(w_2)$ for all $w_1, w_2 \in \mathbb{C}$, but that $\operatorname{Arg}(w_1 w_2) \neq \operatorname{Arg}(w_1) + \operatorname{Arg}(w_2)$ for all $w_1, w_2 \in \mathbb{C}$. This translates into similar statements for the complex-valued logarithm and the principal value of the logarithm.

Exercise 19.9. Show that $\log(w_1 w_2) = \log w_1 + \log w_2$ for all $w_1, w_2 \in \mathbb{C} \setminus \{0\}$. Find values $w_1, w_2 \in \mathbb{C} \setminus \{0\}$ such that $\operatorname{Log}(w_1 w_2) \neq \operatorname{Log} w_1 + \operatorname{Log} w_2$.