Lecture #17: Applications of the Cauchy-Riemann Equations

Example 17.1. Prove that if r and θ are polar coordinates, then the functions $r^n \cos(n\theta)$ and $r^n \sin(n\theta)$ (where n is a positive integer) are harmonic as functions of x and y.

Solution. Consider $r^n \cos(n\theta)$ and $r^n \sin(n\theta)$ where n is a positive integer. The key observation is that de Moivre's formula tells us these are the real and imaginary parts, respectively, of $(r \cos\theta + ir \sin\theta)^n$; that is, if $z = x + iy = re^{i\theta}$, then

$$z^n = r^n e^{in\theta} = r^n \cos(n\theta) + ir^n \sin(n\theta).$$

Hence, let $u = r^n \cos(n\theta)$ and $v = r^n \sin(n\theta)$. In order to show that u and v are harmonic as functions of x and y, we can use Example 14.1 which tells us that the real and imaginary parts of an analytic function are harmonic (assuming the partial derivatives are smooth enough).

Therefore, we see that <u>**if**</u> we can show that $f(z) = z^n$ is analytic, we can conclude for free from Example 14.1 that $u = r^n \cos(n\theta)$ and $v = r^n \sin(n\theta)$ are harmonic as functions of x and y.

In order to prove that $f(z) = z^n$ is analytic, however, we need to show that $f'(z_0)$ exists for all $z_0 \in \mathbb{C}$. Consider

$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \lim_{\Delta z \to 0} \frac{(z_0 + \Delta z)^n - z_0^n}{\Delta z}.$$

By the binomial theorem,

$$(z_0 + \Delta z)^n = \sum_{j=0}^{n} \binom{n}{j} z_0^{n-j} (\Delta z)^j = z_0^n + n z_0^{n-1} \Delta z + \sum_{j=2}^{n} \binom{n}{j} z_0^{n-j} (\Delta z)^j,$$

and so

$$\frac{(z_0 + \Delta z)^n - z_0^n}{\Delta z} = n z_0^{n-1} + \sum_{j=2}^{n} \binom{n}{j} z_0^{n-j} (\Delta z)^{j-1}.$$

Since $j - 1 \geq 0$ for $2 \leq j \leq n$, we immediately deduce that

$$\lim_{\Delta z \to 0} \frac{(z_0 + \Delta z)^n - z_0^n}{\Delta z} = \lim_{\Delta z \to 0} \left[n z_0^{n-1} + \sum_{j=2}^{n} \binom{n}{j} z_0^{n-j} (\Delta z)^{j-1} \right] = n z_0^{n-1}$$

proving $f(z) = z^n$ is entire with $f'(z_0) = n z_0^{n-1}$ for all $z_0 \in \mathbb{C}$. In particular, $u = \operatorname{Re}(z^n) = r^n \cos(n\theta)$ and $v = \operatorname{Im}(z^n) = r^n \sin(n\theta)$ are both harmonic as functions of x and y.

The Cauchy-Riemann Equations and Laplace's Equation in Polar Coordinates

An equivalent way to solve Example 17.1 is to compute $u_{xx} + u_{yy}$ and $v_{xx} + v_{yy}$ directly for both $u = r^n \cos(n\theta)$ and $v = r^n \sin(n\theta)$. The difficulty with this approach is that u and v, as written, are functions of r and θ, but the partials that we wish to compute are with respect to x and y. Therefore, we must use the multivariable chain rule to determine u_r, u_θ, v_r, v_θ in terms of u_x, u_y, v_x, v_y. That is, we will introduce a change of variables

$$U(r, \theta) = u(x, y) \quad \text{and} \quad V(r, \theta) = v(x, y)$$

with $x = r\cos\theta$ and $y = r\sin\theta$. Observe that $r^2 = x^2 + y^2$ so that $2rr_x = 2x$ which implies

$$r_x = \frac{x}{r} = \frac{r\cos\theta}{r} = \cos\theta.$$

Moreover, $\tan\theta = y/x$ so that $\sec^2\theta \cdot \theta_x = -y/x^2$ which implies

$$\theta_x = -\frac{y}{x^2\sec^2\theta} = -\frac{y\cos^2\theta}{x^2} = -\frac{r\sin\theta\cos^2\theta}{r^2\sin^2\theta} = -\frac{\sin\theta}{r}.$$

Similarly,

$$r_y = \sin\theta \quad \text{and} \quad \theta_y = \frac{\cos\theta}{r}.$$

By the chain rule, we now find

$$u_x = U_r r_x + U_\theta \theta_x = (\cos\theta)U_r + (-r^{-1}\sin\theta)U_\theta, \quad u_y = U_r r_y + U_\theta \theta_y = (\sin\theta)U_r + (r^{-1}\cos\theta)U_\theta,$$

and

$$v_x = V_r r_x + V_\theta \theta_x = (\cos\theta)V_r + (-r^{-1}\sin\theta)V_\theta, \quad v_y = V_r r_y + V_\theta \theta_y = (\sin\theta)V_r + (r^{-1}\cos\theta)V_\theta.$$

If we now assume that $f(z) = u(z) + iv(z) = U(r, \theta) + iV(r, \theta)$ is differentiable at $z_0 = r_0 e^{i\theta_0}$ so that the Cauchy-Riemann equations are satisfied at z_0, then

$$u_x(z_0) = v_y(z_0) \quad \text{and} \quad u_y(z_0) = -v_x(z_0).$$

This implies

$$(\cos\theta_0)U_r(r_0, \theta_0) - (r_0^{-1}\sin\theta_0)U_\theta(r_0, \theta_0) = (\sin\theta_0)V_r(r_0, \theta_0) + (r_0^{-1}\cos\theta_0)V_\theta(r_0, \theta_0) \qquad (*)$$

and

$$(\sin\theta_0)U_r(r_0, \theta_0) + (r_0^{-1}\cos\theta_0)U_\theta(r_0, \theta_0) = -(\cos\theta_0)V_r(r_0, \theta_0) + (r_0^{-1}\sin\theta_0)V_\theta(r_0, \theta_0). \qquad (**)$$

Simplifying $(*)$ and $(**)$ yields

$$(U_r(r_0, \theta_0) - r_0^{-1}V_\theta(r_0, \theta_0))\cos\theta_0 - (V_r(r_0, \theta_0) + r_0^{-1}U_\theta(r_0, \theta_0))\sin\theta_0 = 0 \qquad (\dagger)$$

and

$$(V_r(r_0, \theta_0) + r_0^{-1}U_\theta(r_0, \theta_0))\cos\theta_0 + (U_r(r_0, \theta_0) - r_0^{-1}V_\theta(r_0, \theta_0))\sin\theta_0 = 0. \qquad (\ddagger)$$

If we then multiple (†) by $\cos\theta_0$ and (‡) by $\sin\theta_0$, and then add, we obtain

$$(U_r(r_0,\theta_0) - r_0^{-1}V_\theta(r_0,\theta_0))(\cos^2\theta_0 + \sin^2\theta_0) = 0$$

which implies $U_r(r_0,\theta_0) = r_0^{-1}V_\theta(r_0,\theta_0)$. On the other hand, if we then multiple (†) by $-\sin\theta_0$ and (‡) by $\cos\theta_0$, and then add, we obtain

$$(V_r(r_0,\theta_0) + r_0^{-1}U_\theta(r_0,\theta_0))(\cos^2\theta_0 + \sin^2\theta_0) = 0$$

which implies $r_0^{-1}U_\theta(r_0,\theta_0) = -V_r(r_0,\theta_0)$.

Theorem 17.2. *Let $z = re^{i\theta}$. If $f(re^{i\theta}) = U(r,\theta) + iV(r,\theta)$ is differentiable at $z_0 = r_0e^{i\theta_0}$, then the Cauchy-Riemann equations in polar coordinates are satisfied at z_0; that is,*

$$\frac{\partial U}{\partial r}(r_0,\theta_0) = \frac{1}{r_0}\frac{\partial V}{\partial \theta}(r_0,\theta_0) \quad \textit{and} \quad \frac{1}{r_0}\frac{\partial U}{\partial \theta}(r_0,\theta_0) = -\frac{\partial V}{\partial r}(r_0,\theta_0).$$

Summary. The Cauchy-Riemann equations in polar coordinates can be remembered as

$$\boxed{U_r = \frac{1}{r}V_\theta \quad \text{and} \quad \frac{1}{r}U_\theta = -V_r.}$$

Example 17.3. Suppose that $U(r,\theta) = r^n\cos(n\theta)$ and $V(r,\theta) = r^n\sin(n\theta)$. We find

$$\begin{aligned} U_r &= nr^{n-1}\cos(n\theta) \\ V_\theta &= nr^n\cos(n\theta) \end{aligned}$$

and

$$\begin{aligned} U_\theta &= -nr^n\sin(n\theta) \\ V_r &= nr^{n-1}\sin(n\theta) \end{aligned}$$

so that $U_r = r^{-1}V_\theta$ and $r^{-1}U_\theta = -V_r$. Hence, U and V satisfy the Cauchy-Riemann equations in polar coordinates.

We can now use the Cauchy-Riemann equations to derive Laplace's equation in polar coordinates. (Assume that all second partials exist and are sufficiently smooth so that the mixed partials are equal.) That is, we know

$$u_x = v_y \quad \text{implies} \quad rU_r = V_\theta \quad \text{and} \quad u_y = -v_x \quad \text{implies} \quad U_\theta = -rV_r$$

and so taking derivatives with respect to x of the first equation and derivatives with respect to y of the second equation implies

$$0 = (u_x - v_y)_x + (u_y + v_x)_y = (rU_r - V_\theta)_x + (U_\theta + rV_r)_y.$$

Now, using the chain rule, we find

$$(rU_r - V_\theta)_x = r_xU_r + r(U_{rr}r_x + U_{\theta r}\theta_x) - (V_{\theta\theta}\theta_x + V_{r\theta}r_x)$$

and

$$(U_\theta + rV_r)_y = (U_{\theta\theta}\theta_y + U_{r\theta}r_y) + r_y V_r + r(V_{rr}r_y + V_{\theta r}\theta_y).$$

Adding the previous two terms, using the equality of the mixed partials, and simplifying implies

$$r_x U_r + r r_x U_{rr} + (r\theta_x + r_y)U_{\theta r} + \theta_y U_{\theta\theta} = -r_y V_r - r r_y V_{rr} - (r\theta_y - r_x)V_{r\theta} + \theta_x V_{\theta\theta}. \qquad (*)$$

The next step is to note that

$$r\theta_x + r_y = r \cdot -\frac{\sin\theta}{r} + \sin\theta = 0 \quad \text{and} \quad r\theta_y - r_x = r \cdot \frac{\cos\theta}{r} - \cos\theta = 0.$$

so that $(*)$ becomes

$$r_x U_r + r r_x U_{rr} + \theta_y U_{\theta\theta} = -r_y V_r - r r_y V_{rr} + \theta_x V_{\theta\theta}.$$

Substituting in r_x, θ_x, r_y, θ_y, we conclude

$$\cos\theta \left[U_r + rU_{rr} + \frac{1}{r}U_{\theta\theta} \right] = -\sin\theta \left[V_r + rV_{rr} + \frac{1}{r}V_{\theta\theta} \right]. \qquad (\dagger)$$

If, instead, at the beginning of the derivation we had taken derivatives with respect to y of the first equation and derivatives with respect to x of the second equation, then we would have found

$$\cos\theta \left[V_r + rV_{rr} + \frac{1}{r}V_{\theta\theta} \right] = -\sin\theta \left[U_r + rU_{rr} + \frac{1}{r}U_{\theta\theta} \right]. \qquad (\ddagger)$$

We now multiple $(\dagger)$ by $\cos\theta$, multiply $(\ddagger)$ by $\sin\theta$, and add, then we conclude

$$(\cos^2\theta + \sin^2\theta) \left[U_r + rU_{rr} + \frac{1}{r}U_{\theta\theta} \right] = 0$$

and so we finally arrive at Laplace's equation in polar coordinates

$$\boxed{U_{rr} + \frac{1}{r}U_r + \frac{1}{r^2}U_{\theta\theta} = 0.}$$

Note that we can also conclude immediately that V satisfies Laplace's equation in polar coordinates as well,

$$V_{rr} + \frac{1}{r}V_r + \frac{1}{r^2}V_{\theta\theta} = 0.$$

Example 17.4. Suppose that $U(r,\theta) = r^n \cos(n\theta)$. We can now show directly that U is harmonic. That is,

$$U_r = nr^{n-1}\cos(n\theta), \quad U_{rr} = n(n-1)r^{n-2}\cos(n\theta), \quad U_\theta = -nr^n \sin(n\theta), \quad U_{\theta\theta} = -n^2 r^n \cos(n\theta)$$

so that

$$\begin{aligned} U_{rr} + \frac{1}{r}U_r + \frac{1}{r^2}U_{\theta\theta} &= n(n-1)r^{n-2}\cos(n\theta) + \frac{1}{r} \cdot nr^{n-1}\cos(n\theta) + \frac{1}{r^2} \cdot -n^2 r^n \cos(n\theta) \\ &= r^{n-2}\cos(n\theta)[n(n-1) + n - n^2] \\ &= 0 \end{aligned}$$