Mathematics 312 (Fall 2013) October 9, 2013
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Lecture #16: Analytic Properties of the Complex Trigonometric Functions

Recall that we can write the real-valued functions $\sin\theta$ and $\cos\theta$ as

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} \quad \text{and} \quad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}.$$

This motivates the following definition.

Definition. The complex-valued functions $\cos z$ and $\sin z$ are defined to be

$$\cos z = \frac{e^{iz}+e^{-iz}}{2} \quad \text{and} \quad \sin z = \frac{e^{iz}-e^{-iz}}{2i}.$$

We now make a couple of important observations.

- The function e^z is periodic with period $2\pi i$ and the function e^{iz} is periodic with period 2π.
- Since e^{iz} and e^{-iz} are both entire functions, the functions $\cos z$ and $\sin z$ are also entire.
- $\sin(z+2\pi k) = \sin z$ and $\cos(z+2\pi k) = \cos z$ for any integer k. This means that the fundamental region for $\cos z$ and $\sin z$ is $\{0 \le \operatorname{Re} z < 2\pi\}$; see Figure 16.1.

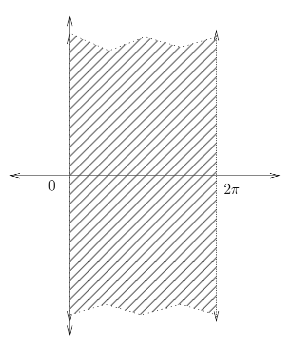

Figure 16.1: The fundamental region for $\cos z$ and $\sin z$.

Example 16.1. Prove that

$$\frac{\mathrm{d}}{\mathrm{d}z}\sin z = \cos z \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}z}\cos z = -\sin z.$$

Solution. We find

$$\frac{\mathrm{d}}{\mathrm{d}z}\sin z = \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{e^{iz}-e^{-iz}}{2i}\right) = \frac{ie^{iz}+ie^{-iz}}{2i} = \frac{e^{iz}+e^{-iz}}{2} = \cos z.$$

and

$$\frac{\mathrm{d}}{\mathrm{d}z}\cos z = \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{e^{iz}+e^{-iz}}{2}\right) = \frac{ie^{iz}-ie^{-iz}}{2} = -\frac{e^{iz}-e^{-iz}}{2i} = -\sin z.$$

The other complex-valued trigonometric functions are defined in the same way as their real counterparts. That is,

- $\tan z = \dfrac{\sin z}{\cos z}$,
- $\sec z = \dfrac{1}{\cos z}$,
- $\csc z = \dfrac{1}{\sin z}$, and
- $\cot z = \dfrac{1}{\tan z} = \dfrac{\cos z}{\sin z}$.

Note that $\cot z$ and $\csc z$ are analytic except at the zeroes of $\sin z$, namely at $z = k\pi$, $k \in \mathbb{Z}$. Also note that $\tan z$ and $\sec z$ are analytic except at the zeroes of $\cos z$, namely at $z = \pi/2 + k\pi$, $k \in \mathbb{Z}$.

Exercise 16.2. Show that the following identities hold for complex variables z, z_1, and z_2:

- $\sin(-z) = -\sin z$, $\cos(-z) = \cos z$,
- $\sin^2 z + \cos^2 z = 1$,
- $\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \sin z_2 \cos z_1$,
- $\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_2 \sin z_1$,
- $\sin(2z) = 2\sin z \cos z$, $\cos(2z) = \cos^2 z - \sin^2 z$.

In fact, we can also show that the differentiation formulas that hold for real-valued trigonometric functions also hold for the complex-valued ones; that is,

- $\dfrac{\mathrm{d}}{\mathrm{d}z}\tan z = \sec^2 z$,
- $\dfrac{\mathrm{d}}{\mathrm{d}z}\cot z = -\csc^2 z$,
- $\dfrac{\mathrm{d}}{\mathrm{d}z}\sec z = \sec z \tan z$, and
- $\dfrac{\mathrm{d}}{\mathrm{d}z}\csc z = -\csc z \cot z$.

Exercise 16.3. Verify the previous differentiation formulas hold.

We end this lecture with one final set of definitions. The complex-valued *hyperbolic cosine* function is defined to be

$$\cosh(z) = \frac{e^z + e^{-z}}{2},$$

and the complex-valued *hyperbolic sine* function is defined to be

$$\sinh(z) = \frac{e^z - e^{-z}}{2}.$$

Note that

$$\cosh(iz) = \cos(z) \quad \text{and} \quad -i\sinh(iz) = \sin(z).$$

Finally, the complex-valued *hyperbolic tangent* function is defined to be

$$\tanh(z) = \frac{\sinh(z)}{\cosh(z)} = \frac{e^z - e^{-z}}{e^z + e^{-z}}.$$

Exercise 16.4. Show that the following identities hold for the complex hyperbolic trigonometric functions:

- $\cosh(z) = \cos(iz)$,
- $\sinh(z) = -i\sin(iz)$,
- $\cosh^2(z) - \sinh^2(z) = 1$,
- $\frac{\mathrm{d}}{\mathrm{d}z}\cosh(z) = \sinh(z)$,
- $\frac{\mathrm{d}}{\mathrm{d}z}\sinh(z) = \cosh(z)$.