Mathematics 312 (Fall 2013) October 7, 2013

Prof. Michael Kozdron

Lecture #15: Analytic Properties of the Complex Exponential

Recall from Lecture #13 that we set out to determine when a function is differentiable. One consequence of our calculations was the following. We showed that if f was differentiable at z_0, then f satisfied the Cauchy-Riemann equations at z_0. The way we derived this result was to compute $f'(z_0)$ in two ways and then equate real and imaginary parts. If we step back, however, we can view our computations as a way of calculating $f'(z_0)$.

Theorem 15.1. *Consider the function $f(z) = u(z) + iv(z)$ defined in some neighbourhood of z_0. If f is differentiable at $z_0 = x_0 + iy_0$, then*

$$f'(z_0) = \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0)$$

and

$$f'(z_0) = \frac{\partial v}{\partial y}(x_0, y_0) - i\frac{\partial u}{\partial y}(x_0, y_0).$$

Remark. It is important to stress that we must still know *a priori* that f is differentiable at z_0 in order to conclude that its derivative is given by either of these formulas. The most common way of doing this is to use Theorem 14.3.

Example 15.2. Consider the complex exponential function

$$f(z) = e^z = e^x e^{iy} = e^x[\cos y + i \sin y].$$

Use Theorem 14.3 to show that $f(z)$ is entire, and then use Theorem 15.1 to compute $f'(z_0)$ for every $z_0 \in \mathbb{C}$. Also show that $f(\mathbb{C}) = \mathbb{C} \setminus \{0\}$.

Solution. If $f(z) = e^z = e^x e^{iy} = e^x[\cos y + i \sin y]$, then

$$\frac{\partial u}{\partial x}(x_0, y_0) = e^{x_0} \cos y_0, \quad \frac{\partial v}{\partial x}(x_0, y_0) = e^{x_0} \sin y_0,$$

and

$$\frac{\partial v}{\partial y}(x_0, y_0) = e^{x_0} \cos y_0, \quad \frac{\partial u}{\partial y}(x_0, y_0) = -e^{x_0} \sin y_0.$$

Observe that

$$\frac{\partial u}{\partial x}(z_0), \quad \frac{\partial u}{\partial y}(z_0), \quad \frac{\partial v}{\partial x}(z_0), \quad \frac{\partial v}{\partial y}(z_0)$$

exist for all $z_0 \in \mathbb{C}$ and are clearly continuous at z_0. Since the Cauchy-Riemann equations are also satisfied for every $z_0 \in \mathbb{C}$, we conclude from Theorem 14.3 that $f(z) = e^z$ is differentiable at every $z_0 \in \mathbb{C}$. Hence, e^z is necessarily analytic at every $z_0 \in \mathbb{C}$ so that e^z is entire. We can now apply Theorem 15.1 to conclude

$$f'(z_0) = \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0) = e^{x_0} \cos y_0 + ie^{x_0} \sin y_0 = e^{z_0}$$

for every $z_0 \in \mathbb{C}$.

Observe that if $z \in \mathbb{C}$, then $e^z \neq 0$. This follows from the fact that $e^x > 0$ for every $x \in \mathbb{R}$ and $\cos y + i \sin y \neq 0$ for every $y \in \mathbb{R}$ (i.e., $\cos y$ and $\sin y$ are never simultaneously equal to 0). To finish the proof that $f(\mathbb{C}) = \mathbb{C} \setminus \{0\}$, suppose that $w \in \mathbb{C} \setminus \{0\}$ and observe that

$$e^{\log |w|}(\cos(\operatorname{Arg} w) + i \sin(\operatorname{Arg} w)) = w.$$

In other words, if $z = \log |w| + i \operatorname{Arg} w$, then

$$e^z = e^{\log |w| + i \operatorname{Arg} w} = |w| e^{i \operatorname{Arg}(w)} = w.$$

Since $\cos y$ and $\sin y$ are periodic with period 2π, we conclude that

$$e^z = e^{z+2\pi i}.$$

That is, e^z is periodic with period $2\pi i$. Since $\operatorname{Arg}(w) \in (-\pi, \pi]$, we therefore take the *fundamental region* for e^z to be

$$\{-\pi < \operatorname{Im} z \leq \pi\}$$

as shown in Figure 15.1.

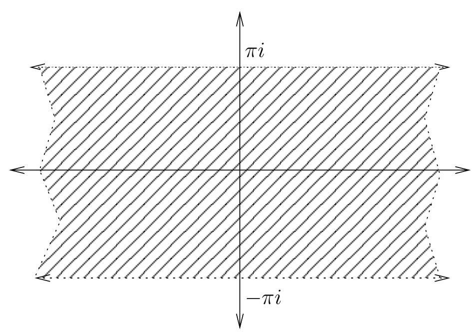

Figure 15.1: The fundamental region for e^z.

In fact, this is what motivates the definition of the complex logarithm function. Let $w \in \mathbb{C}$ with $w \neq 0$. We know that there are infinitely many values of $z \in \mathbb{C}$ such that $e^z = w$; see Figure 15.2.

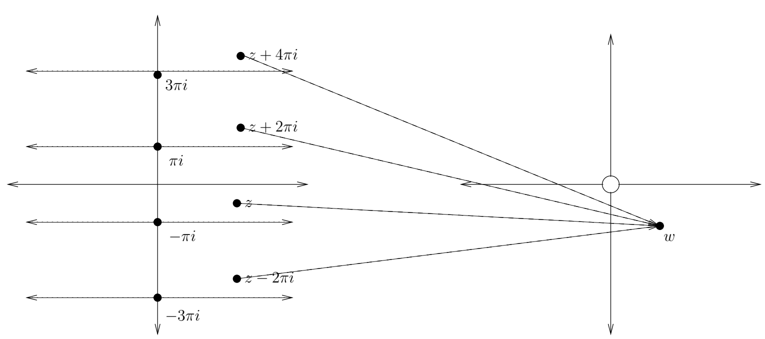

Figure 15.2: The image of $\mathbb{C}$ under the mapping e^z.

However, there is a *unique* value of z in the fundamental region $\{-\pi < \operatorname{Im} z \leq \pi\}$ with $e^z = w$. This is what we will use to define the logarithm of w; more precisely, this will be the *principal value of the logarithm*.

Definition. Suppose that $w \in \mathbb{C} \setminus \{0\}$. We define the *principal value of the logarithm of* w, denoted $\operatorname{Log} w$, to be

$$\operatorname{Log} w = \log |w| + i \operatorname{Arg}(w).$$