Mathematics 312 (Fall 2013) October 4, 2013
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Lecture #14: Harmonicity and the Cauchy-Riemann Equations

Recall from last class that if $f(z) = u(z) + iv(z)$ is analytic in a domain D, then u and v satisfy the Cauchy-Riemann equations in D, namely

$$u_x(z_0) = v_y(z_0) \quad \text{and} \quad u_y(z_0) = -v_x(z_0)$$

for every $z_0 = x_0 + iy_0 \in D$.

Example 14.1. Suppose that $f = u + iv$ is analytic in a domain D. Show that u satisfies *Laplace's equation* in D (assuming that u_{xx}, u_{yy}, v_{xy}, v_{yx} exist in D and are sufficiently smooth so that $v_{xy} = v_{yx}$). Next show that v also satisfies Laplace's equation in D (assuming that v_{xx}, v_{yy}, u_{xy}, u_{yx} exist in D and are sufficiently smooth so that $u_{xy} = u_{yx}$).

Solution. Since $f(z) = u(z) + iv(z) = u(x, y) + iv(x, y)$ is analytic in D, we know the Cauchy-Riemann equations are satisfied at any $z_0 = x_0 + iy_0 \in D$. This means that

$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0) \quad \text{and} \quad \frac{\partial v}{\partial x}(x_0, y_0) = -\frac{\partial u}{\partial y}(x_0, y_0).$$

Taking the second partials of u with respect to x and y implies that

$$\frac{\partial^2 u}{\partial x^2}(x_0, y_0) = \frac{\partial^2 v}{\partial x \partial y}(x_0, y_0) \quad \text{and} \quad \frac{\partial^2 v}{\partial y \partial x}(x_0, y_0) = -\frac{\partial^2 u}{\partial y^2}(x_0, y_0)$$

and so

$$\frac{\partial^2 u}{\partial x^2}(x_0, y_0) + \frac{\partial^2 u}{\partial y^2}(x_0, y_0) = \frac{\partial^2 v}{\partial x \partial y}(x_0, y_0) - \frac{\partial^2 v}{\partial y \partial x}(x_0, y_0) = 0.$$

On the other hand, taking the second partials of v with respect to x and y implies that

$$\frac{\partial^2 v}{\partial y^2}(x_0, y_0) = \frac{\partial^2 u}{\partial y \partial x}(x_0, y_0) \quad \text{and} \quad \frac{\partial^2 v}{\partial x^2}(x_0, y_0) = -\frac{\partial^2 u}{\partial x \partial y}(x_0, y_0)$$

and so

$$\frac{\partial^2 v}{\partial x^2}(x_0, y_0) + \frac{\partial^2 v}{\partial y^2}(x_0, y_0) = -\frac{\partial^2 u}{\partial x \partial y}(x_0, y_0) + \frac{\partial^2 u}{\partial y \partial x}(x_0, y_0) = 0.$$

Definition. Suppose that $D \subseteq \mathbb{C}$ is a domain. We say that a function $u : D \to \mathbb{R}$ is *harmonic* if each of u_{xx}, u_{yy}, u_{xy}, and u_{yx} is continuous in D and if u satisfies Laplace's equation in D, namely

$$u_{xx}(x_0, y_0) + u_{yy}(x_0, y_0) = 0$$

for every $z_0 = x_0 + iy_0 \in D$.

Example 14.2. Suppose that $u : \mathbb{C} \to \mathbb{R}$ is given by $u(z) = u(x, y) = x^3 - 3xy^2 + y$. Verify that u is harmonic in $\mathbb{C}$, and then find an analytic function $f : \mathbb{C} \to \mathbb{C}$ with $\operatorname{Re} f(z) = u(z)$.

Solution. To show that u is harmonic in $\mathbb{C}$, we need to show (i) u_{xx}, u_{yy}, u_{xy}, and u_{yx} are continuous, and (ii) $u_{xx} + u_{yy} = 0$. That is,

$$u_x = 3x^2 - 3y^2 \quad \text{so that} \quad u_{xx} = 6x \quad \text{and} \quad u_{yx} = -6y$$

and

$$u_y = -6xy + 1 \quad \text{so that} \quad u_{yy} = -6x \quad \text{and} \quad u_{xy} = -6y.$$

Clearly, u_{xx}, u_{yy}, u_{xy}, and u_{yx} are continuous and

$$u_{xx} + u_{yy} = 6x - 6x = 0$$

so that u is, in fact, harmonic in $\mathbb{C}$. To find an analytic function f with $\operatorname{Re} f(z) = u(z)$ means that we must find $v(z)$ such that $f(z) = u(z) + iv(z)$ is analytic in $\mathbb{C}$. Note that $v(z)$ is called a *harmonic conjugate* of $u(z)$. (As we will see shortly, $v(z)$ is not unique.) Since f is assumed to be analytic, we know that u and v must satisfy the Cauchy-Riemann equations. That is,

$$u_x = v_y \quad \text{implies} \quad v_y = 3x^2 - 3y^2$$

and

$$u_y = -v_x \quad \text{implies} \quad v_x = 6xy - 1.$$

Integrating v_y implies

$$v(x, y) = 3x^2y - y^3 + C_1(x)$$

and integrating v_x implies that

$$v(x, y) = 3x^2y - x + C_2(y).$$

By comparing these two expressions for $v(x, y)$, we see that $v(x, y)$ must be of the form

$$v(x, y) = 3x^2y - y^3 - x + C$$

where C is an arbitrary real constant. Since the problem asks us to find *one* analytic function f with $\operatorname{Re} f(z) = u(z)$, the one we'll choose is

$$f(z) = f(x, y) = u(x, y) + iv(x, y) = x^3 - 3xy^2 + y + i(3x^2y - y^3 - x + 312).$$

It is worth noting that we can write $f(z)$ as a function of z as follows:

$$f(z) = z^3 - iz + 312i.$$

We end this lecture with a partial converse to the Cauchy-Riemann equations. As we demonstrated last lecture, if we know that $f'(z_0)$ exists, then the Cauchy-Riemann equations are satisfied at z_0. However, as we saw in Exercise 13.7, it is possible for the Cauchy-Riemann equations to be satisfied at a point, yet for the function not to be differentiable at that point. The following theorem, whose proof may be found on pages 74–76 of the text by Saff and Snider, gives a sufficient condition for a function to be differentiable at z_0 in terms of the Cauchy-Riemann equations.

Theorem 14.3. *Let $f(z)$ be defined in some neighbourhood D of the point $z_0 = x_0 + iy_0$. If the Cauchy-Riemann equations are satisfied at z_0, namely*

$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0) \quad \text{and} \quad \frac{\partial v}{\partial x}(x_0, y_0) = -\frac{\partial u}{\partial y}(x_0, y_0),$$

and if

$$\frac{\partial u}{\partial x}, \quad \frac{\partial u}{\partial y}, \quad \frac{\partial v}{\partial x}, \quad \frac{\partial v}{\partial y}$$

all exist in D and are continuous at z_0, then f is differentiable at z_0.

Definition. An *entire* function is one that is analytic in the entire complex plane.